J. DIFFERENTIAL GEOMETRY
47 (1997) 560-599

CASSON INVARIANT, SIGNATURE DEFECT OF FRAMED MANIFOLDS AND THE SECONDARY CHARACTERISTIC CLASSES OF SURFACE BUNDLES

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1. Introduction

The purpose of the present paper is twofold. First we relate the Casson invariant to the signature defect of Hirzebruch, which is a fine invariant defined for framed manifolds. Secondly we introduce the notion of the *secondary* characteristic classes of surface bundles. This consists of a series of cohomology classes of a certain subgroup of the mapping class group of orientable surfaces. The definition is given by first examining the above relation between the Casson invariant and the signature defect precisely and then generalizing it to higher dimensional context.

The Casson invariant is an integer valued invariant defined for oriented homology 3-spheres. Roughly speaking it counts the number (with signs) of conjugacy classes of irreducible representations of the fundamental group of an oriented homology 3-sphere into the Lie group SU(2). In the cases where the homology 3-spheres are given as the results of performing $1/n$ Dehn surgeries along knots in the 3-dimensional sphere S^3, there is a formula which expresses their Casson invariants in terms of a classical invariant of knots, namely the Alexander polynomial.

In our previous papers [22], [23], we investigated how the Casson invariant is related to the structure of the mapping class group of oriented surfaces through the correspondence between elements of the mapping class group and 3-manifolds via the Heegaard splittings. Here we briefly

Received March 8, 1995, and, in revised form, July 9, 1997.

recall our results. Let $\mathcal{M}_g$ be the mapping class group of an oriented closed surface of genus g, which we denote by Σ_g, and let $\mathcal{K}_g$ be the subgroup of $\mathcal{M}_g$ generated by all the Dehn twists along *separating* simple closed curves on Σ_g. Let us fix a Heegaard splitting of S^3 of genus g. Namely we express $S^3 = H_g \cup_{\iota_g} -H_g$ where H_g denotes the handlebody of genus g, and $\iota_g \in \mathcal{M}_g$ is a certain element so that if we glue H_g with $-H_g$ along their boundaries by the map ι_g, then we obtain S^3. Thus to each element $\varphi \in \mathcal{K}_g$, we can associate an oriented homology 3-sphere W_φ which is the one obtained by first cutting S^3 along the fixed Heegaard surface Σ_g and then reglueing the resulting two pieces by the map φ. Namely we have $W_\varphi = H_g \cup_{\iota_g \varphi} -H_g$. It can be shown that any homology 3-sphere can be obtained in this way, for some g and $\varphi \in \mathcal{K}_g$ (see [22]). Then we can define a mapping

$$\lambda^* : \mathcal{K}_g \longrightarrow \mathbf{Z}$$

by setting $\lambda^*(\varphi) = \lambda(W_\varphi)(\varphi \in \mathcal{K}_g)$ where $\lambda(W_\varphi)$ is the Casson invariant of the homology sphere W_φ. In such a situation, the main results of [22] can be summarized as follows. Let $\mathcal{M}_{g,1}$ be the mapping class group of Σ_g *relative* to an embedded disc $D^2 \subset \Sigma_g$, and let $\mathcal{K}_{g,1}$ be the subgroup of $\mathcal{M}_{g,1}$ generated by all the Dehn twists along separating simple closed curves on $\Sigma_g \backslash D^2$. Then we can define two homomorphisms $d : \mathcal{K}_{g,1} \to \mathbf{Z}$ and $q : \mathcal{K}_{g,1} \to \mathbf{Q}$ such that the equality

$$\lambda^*(\varphi) = \frac{1}{24} d(\widetilde{\varphi}) + q(\widetilde{\varphi})$$

holds for all $\varphi \in \mathcal{K}_g$ where $\widetilde{\varphi} \in \mathcal{K}_{g,1}$ is any lift of φ. In other words, the Casson invariant λ^* can be described as the sum of two invariants d and q which are defined for elements of $\mathcal{K}_{g,1}$. The latter invariant q is rather easy to compute and it should measure certain higher order Massey products of the mapping torus $M_\varphi = \Sigma_g \times I/\sim$, where we identify $(p, 0)$ with $(\varphi(p), 1)$ $(p \in \Sigma_g)$ and M_φ homologically the same as $\Sigma_g \times S^1$. On the contrary, d is a much deeper invariant than q, and in fact it plays the role of the very core of the Casson invariant from our point of view. It is true that the value of d on each generator of the group $\mathcal{K}_{g,1}$, namely Dehn twist along any separating simple closed curve on $\Sigma_g \setminus D^2$, is very simple. However at present we do not know any effective method to compute it for arbitrarily given element of $\mathcal{K}_{g,1}$, because there is no algorithm to express any given element as a product of generators. The situation being like this, we must say that d remains to be a rather mysterious invariant.

Now one of the main purposes of the present paper is to give a geometrical interpretation of this invariant d. Precise statements will be described in §5 after a few preparatory sections (§§2-4). In short, we will prove that the value $d(\varphi)$ is equal to the Hirzebruch's *signature defect* (see [14]) of the mapping torus M_φ with respect to a certain canonical framing of its tangent bundle (Theorem 5.1). In this way we found a connection between the Casson invariant and the signature defect of certain framed 3-manifolds.

The invariant d is defined not just for elements of the group $\mathcal{K}_{g,1}$ but also for any element of the whole mapping class group $\mathcal{M}_{g,1}$ (see [22]). Our geometrical interpretation of the invariant d mentioned above is in fact given in this general setting. It takes the form of the signature defect of the mapping torus $M_\varphi(\varphi \in \mathcal{M}_{g,1})$ with respect to a certain *2-framing* (in the sense of Atiyah [4]) of its tangent bundle (Theorem 5.2). Our 2-framings of these mapping tori M_φ are different from Atiyah's *canonical* 2-framings, and here we find an interesting interplay between Atiyah's argument in [4] and ours. Proofs of these results will be given in §6.

In §7 we generalize some of the above arguments given mainly for the 3-dimensional manifolds M_φ, to the cases of higher dimensions. In particular we prove that any Σ_g-bundle whose holonomy group is contained in the subgroup $\mathcal{K}_{g,1}$ has a *partial framing* on its tangent bundle. We can generalize the definition of the signature defect so that it will be adapted to this weaker setting (Proposition 7.3). By making use of this together with the bordism theory as developed in [1], [8], we obtain the definition of our *secondary* characteristic classes of surface bundles

$$d_i \in H^{4i-3}(\mathcal{K}_{g,1}; \mathbf{Q}) \quad (i = 1, 2, \cdots),$$

(see Theorem 7.7). The first class $d_1 \in H^1(\mathcal{K}_{g,1}; \mathbf{Q})$ is nothing but the above invariant $d : \mathcal{K}_{g,1} \to \mathbf{Z}$, up to a non-zero factor, which embodies a deep structure of the mapping class group related to the Casson invariant. We expect that other secondary characteristic classes of higher degrees will also reflect certain unknown structure of the mapping class group.

In this paper, we treat our invariant d only *locally*, namely we consider its value on each individual element of the mapping class group and give a geometrical meaning to it. In future, we would like to investigate this invariant as well as higher secondary classes more *globally*, perhaps in relation to the global geometry of the Teichmüller space and also of various universal families over it. Here Atiyah's work [3], which treats the case of genus one, should be considered as a guide.

2. Cocycles for the first characteristic class of surface bundles

Let Σ_g be a closed oriented surface of genus g and let $\mathcal{M}_g$ be the mapping class group of Σ_g. Namely it is the group of all the isotopy classes of orientation preserving diffeomorphisms of Σ_g. Harer [12] proved that the second cohomology group $H^2(\mathcal{M}_g;\mathbf{Z})$ of $\mathcal{M}_g$ is an infinite cyclic group (for $g \geq 3$) generated by the first Chern class

$$c_1 \in H^2(\mathcal{M}_g;\mathbf{Z}).$$

Here c_1 is defined as follows. Let $\rho : \mathcal{M}_g \to \mathrm{Sp}(2g;\mathbf{Z})$ be the classical representation of $\mathcal{M}_g$ onto the Siegel modular group which is induced from the natural action of $\mathcal{M}_g$ on the integral homology group $H_1(\Sigma_g;\mathbf{Z})$ of Σ_g. The Siegel modular group is contained in the symplectic group $\mathrm{Sp}(2g;\mathbf{R})$, and the maximal compact subgroup of the latter group is isomorphic to the unitary group $\mathrm{U}(g)$. Hence we have the usual first Chern class $c_1 \in H^2(\mathrm{BSp}(2g;\mathbf{R});\mathbf{Z})$. This cohomology class can be pulled back to define an element in $H^2(\mathrm{Sp}(2g;\mathbf{Z});\mathbf{Z})$ and then in $H^2(\mathcal{M}_g;\mathbf{Z})$ via the homomorphism ρ. This is our first Chern class c_1 (we use the same letter). In the case where we are given a family of Riemann surfaces of genus g over a differentiable manifold M, our c_1 is (up to signs) nothing but the first Chern class of the g-dimensional complex vector bundle over M whose fibre over a point $p \in M$ is the space of holomorphic differentials of the Riemann surface over p (cf. [2], [19]).

Now let

$$e_1 \in H^2(\mathcal{M}_g;\mathbf{Z})$$

be the first characteristic class of surface bundles defined in [18], [19] (see also [2], [17], [27]). Here we briefly recall its definition. Let $\pi : E \to M$ be a differentiable fibre bundle over M with fibre Σ_g, and let ξ be the tangent bundle along the fibres of π, namely it is the subbundle of the tangent bundle of E consisting of all vectors which are tangent to the fibres. We assume that π is *oriented* in the sense that ξ is an oriented plane bundle over E. Let $e \in H^2(E;\mathbf{Z})$ be the Euler class of ξ, and set

$$e_1(\pi) = \pi_*(e^2) \in H^2(M;\mathbf{Z}),$$

where $\pi_* : H^4(E;\mathbf{Z}) \to H^2(M;\mathbf{Z})$ is the Gysin homomorphism or the integration along the fibres. By an obvious naturality of the above construction, we have a well defined cohomology class $e_1 \in H^2(\mathcal{M}_g;\mathbf{Z})$. This is the definition of our first characteristic class e_1. If we apply the Gysin homomorphism to powers of the Euler class e of higher degrees, we obtain higher characteristic classes of surface bundles (see [19], [27] for details).

Now the relation between the two cohomology classes c_1 and e_1 is very simple; namely we have

$$e_1 = 12c_1.$$

This can be proved by applying the Atiyah-Singer index theorem for families of elliptic operators [5] or the Grothendieck Riemann-Roch theorem (cf. [2], [19], [27]). If we assume Harer's result mentioned above, then the above equality holds over the integers.

Now there are various explicit group cocycles which represent (certain multiples) of the first Chern class $c_1 \in H^2(\mathcal{M}_g;\mathbf{Z})$ (see [9], [11], [16], [21], [25], [29]). Except for just one cocycle defined in [21], [25], all of these cocycles are defined at the level of the Siegel modular group $\mathrm{Sp}(2g;\mathbf{Z})$ or the symplectic group $\mathrm{Sp}(2g;\mathbf{R})$, and each of them has its own interesting geometrical or algebraic background, e.g. the volumes of geodesic simplices in the Siegel upper half spaces ([9]), the Maslov indices of Lagrangian subspaces of a symplectic vector space ([6], [11], [29], [30]) and so on.

Among these cocycles, the most important one for our purpose is Meyer's *signature cocycle* [16] which is defined as follows. For any two elements $\varphi, \psi \in \mathcal{M}_g$, let $M_{(\varphi,\psi)}$ be the Σ_g-bundle over the two sphere with three holes (or a pair of pants) such that its boundary is a disjoint union of M_φ, M_ψ and $-M_{\varphi\psi}$. If we put

$$\tau(\varphi,\psi) = \operatorname{sign} M_{(\varphi,\psi)},$$

where $\operatorname{sign} M_{(\varphi,\psi)}$ denotes the signature of the 4-manifold $M_{(\varphi,\psi)}$, then it can be shown that τ is a 2-cocycle of the group $\mathcal{M}_g$ and represents $4c_1 \in H^2(\mathcal{M}_g;\mathbf{Z})$ (see [3], [16]). It follows that 3τ represents the first characteristic class e_1. Meyer gave an explicit formula for the cocycle τ at the level of the Siegel modular group $\mathrm{Sp}(2g;\mathbf{Z})$.

Next we recall the definition of our group cocycle c which also represents the first characteristic class e_1 (see [21]). At present this is the only known group cocycle that *cannot* be defined at the level of the Siegel modular group. We may say that it reflects deeper structure of the mapping class group than any other cocycles. For technical reasons, this cocycle is defined not on the usual mapping class group $\mathcal{M}_g$ but on another type of mapping class group $\mathcal{M}_{g,1}$ which is the one *relative* to an embedded disc $D^2 \subset \Sigma_g$; namely we consider only those diffeomorphisms and isotopies which are the identity on D^2. (However over the rationals we *do* have a related group cocycle which is defined on $\mathcal{M}_g$; see [25].) We proved in [20] that the cohomology group $H^1(\mathcal{M}_{g,1}; H_1(\Sigma_g; \mathbf{Z}))$ is an infinite cyclic group. Let us fix a crossed homomorphism

$$k : \mathcal{M}_{g,1} \longrightarrow H_1(\Sigma_g; \mathbf{Z})$$

which represents a generator of the above cohomology group. Here a crossed homomorphism means that we have the equality $k(\varphi\psi) = k(\varphi) + \varphi_*(k(\psi))$ for all $\varphi, \psi \in \mathcal{M}_{g,1}$. Now in terms of this crossed homomorphism, our group cocycle c is defined as

$$c(\varphi, \psi) = k(\varphi^{-1}) \cdot k(\psi),$$

where $\cdot$ denotes the intersection number. We mention here that the crossed homomorphism k is well defined up to coboundaries, which can be naturally identified with elements of $H_1(\Sigma_g; \mathbf{Z})$.

Henceforth let us call this cocycle c, which represents the first characteristic class $e_1 \in H^2(\mathcal{M}_{g,1}; \mathbf{Z})$, the *intersection cocycle*.

3. A central extension of the mapping class group and framings of 3-manifolds

In this section, first we recall the notion of 2-framings of 3-manifolds and the definition of a certain central extension of the mapping class group $\mathcal{M}_g$ from Atiyah's paper [4], in which they are fully developed. According to [4], these ideas are all due to Witten. In the same paper, Atiyah clarified the relationship between Meyer's signature cocycle and the above ideas by introducing the notion of *canonical 2-framing* to every 3-manifold. In the latter half of this section, we will consider how our intersection cocycle is related to these ideas.

Let M be a closed oriented 3-manifold. Then it is easy to see that the twice of the tangent bundle of M, denoted by $2TM = TM \oplus TM$, has a natural spin structure. Moreover it is a trivial bundle because TM is trivial. A homotopy class of trivializations of $2TM$ as a Spin(6)-bundle is called a 2-framing of M. The set of 2-framings of any 3-manifold can be naturally identified with the set of integers $\mathbf{Z}$ as follows. Let α be a 2-framing of M. Choose a compact oriented 4-manifold Z such that $\partial Z = M$, and set $\delta(\alpha) = \frac{1}{6}p_1(2TZ; \alpha) - \operatorname{sign} Z$ where the relative Pontrjagin number p_1 is defined by trivializing $2TZ$ on $\partial Z = M$ by the 2-framing α. It is easy to see that $\delta(\alpha)$ is well defined independent of the choice of Z. In fact it is the signature defect of Hirzebruch adapted to the context of 2-framings. Now the identification of the set of 2-framings with $\mathbf{Z}$ mentioned above is given by associating $3\delta(\alpha)$ to each 2-framing α. In these terminologies, Atiyah's canonical 2-framing of a 3-manifold is defined to be the unique 2-framing α such that $\delta(\alpha) = 0$.

Next we recall the definition of a central extension

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widehat{\mathcal{M}}_g \longrightarrow \mathcal{M}_g \longrightarrow 1.$$

As a set, $\widehat{\mathcal{M}}_g$ is the one consisting of isomorphism classes of oriented surface bundles $M \to S^1$ over the circle with fibre Σ_g together with a choice of 2-framing α on the total space M as well as an identification of the fibre over the base point $* \in S^1$ with Σ_g (the definition given in [4] is insufficient. One must fix a trivialization of each surface bundle over the base point as above). Thus any element of $\widehat{\mathcal{M}}_g$ can be represented by a pair (φ, α) where $\varphi \in \mathcal{M}_g$ and α is a 2-framing on the mapping torus M_φ. The group law is defined as $(\varphi, \alpha)(\psi, \beta) = (\varphi\psi, \gamma)$ where γ is the 2-framing on $M_{\varphi\psi}$ such that the relative p_1 of $M_{(\varphi,\psi)}$ with respect to the 2-framing $\alpha + \beta - \gamma$ on $\partial M_{(\varphi,\psi)} = M_\varphi + M_\psi - M_{\varphi\psi}$ vanishes.

Now let $s_0 : \mathcal{M}_g \to \widehat{\mathcal{M}}_g$ be the canonical section to the central extension $\mathbf{Z} \to \widehat{\mathcal{M}}_g \to \mathcal{M}_g$ given by $s_0(\varphi) = (\varphi, \alpha_\varphi)$ $(\varphi \in \mathcal{M}_g)$, where α_φ is Atiyah's canonical 2-framing on M_φ. Then it turns out that the associated 2-cocycle of the above extension

$$\mathcal{M}_g \ni \varphi, \psi \mapsto s_0(\varphi)s_0(\psi)\{s_0(\varphi\psi)\}^{-1} \in \mathbf{Z}$$

is equal to thrice Meyer's signature cocycle τ. It follows that the extension class of the above central extension is $12c_1$ and hence is nothing but the first characteristic class $e_1 \in H^2(\mathcal{M}_g; \mathbf{Z})$.

Now we would like to investigate the influence of our intersection cocycle c, which also represents e_1, on the above central extension. But before doing so, we make a short digression to consider a general fact about central extensions which should be well known.

Geometrically a central extension corresponds to a principal SO(2)-bundle or equivalently an oriented S^1-bundle. To mesure how such a bundle is twisted globally, we put a *connection* on it which is a 1-form on the total space satisfying certain conditions. We then have the associated *curvature* form which is a closed 2-form on the base space. The de Rham cohomology class of this closed 2-form is nothing but the Euler class of the given bundle (up to the factor 2π). Another method to compute the Euler class is to try to make a cross section to the associated oriented plane bundle whose image is transverse to the zero-section. Then the intersection of the image of the section with the zero-section represents the Poincaré dual of the Euler class. Now let us consider how these geometrical ideas can be converted into the algebraic context of central extensions of groups.

Let

$$0 \longrightarrow \mathbf{Z} \longrightarrow \hat{\Gamma} \longrightarrow \Gamma \longrightarrow 1$$

be a central extension and let $\chi \in H^2(\Gamma; \mathbf{Z})$ be its Euler class. A *section* of this central extension is a set map $s : \Gamma \to \hat{\Gamma}$ which is a right inverse to the projection $\pi : \hat{\Gamma} \to \Gamma$. A *retraction* is a mapping $r : \hat{\Gamma} \to \mathbf{Z}$ such that the equality $r(\hat{f} + m) = r(\hat{f}) + m$ holds for any $\hat{f} \in \hat{\Gamma}$ and $m \in \mathbf{Z}$ where we write the multiplication by $m \in \mathbf{Z}$ additively. A *cocycle* of this central extension is a group 2-cocycle $c \in Z^2(\Gamma; \mathbf{Z})$ which represents the Euler class χ. In these terminologies, we have

Proposition 3.1. *Let $0 \to \mathbf{Z} \to \hat{\Gamma} \to \Gamma \to 1$ be a central extension. Then there is a natural bijection between the following two sets:*

(i) the set of all sections,

(ii) the set of all retractions.

Moreover if we assume that $H^1(\Gamma; \mathbf{Z}) = 0$, then there is also a canonical bijection between any of the above two sets and

(iii) the set of all cocycles.

The normalizing condition (i) $s(\mathrm{id}) = \mathrm{id}$ for elements of the first set corresponds to the conditions (ii) $r(\mathrm{id}) = 0$ and (iii) $c(f, \mathrm{id}) = c(\mathrm{id}, f) = 0$ for any $f \in \Gamma$, for elements of the other two sets respectively, under the above bijections.

Proof. The natural one-to-one correspondence between the two sets (i) and (ii) is given as follows. To each section $s : \Gamma \to \hat{\Gamma}$ we associate a retraction $r : \hat{\Gamma} \to \mathbf{Z}$ such that $r(\hat{f}) = \hat{f}s(f)^{-1}$ where $\hat{f} \in \hat{\Gamma}$ is any element and $f = \pi(\hat{f})$. Conversely, if we are given a retraction $r : \hat{\Gamma} \to \mathbf{Z}$, then we can construct a section $s : \Gamma \to \hat{\Gamma}$ by setting $s(f)(f \in \Gamma)$ to be the unique element $\hat{f} \in \hat{\Gamma}$ such that $\pi(\hat{f}) = f$ and $r(\hat{f}) = 0$. It is easy to see that the above two correspondences are mutually inverses.

Next we consider the relation between the two sets (i) and (iii). Any section $s : \Gamma \to \hat{\Gamma}$ gives rise to a cocycle c which is defined by

$$c(f,g) = s(f)s(g)\{s(fg)\}^{-1}(f,g \in \Gamma).$$

For later use, let us fix a section s_0 and let c_0 be the associated cocycle. Now let c be any cocycle. Both of the cocycles c_0 and c represent the same cohomology class, namely the Euler class of the given central extension. Hence there exists a 1-cochain $d : \Gamma \to \mathbf{Z}$ such that $c = c_0 + \delta d$. If we define a section $s : \Gamma \to \hat{\Gamma}$ by $s = s_0 + d$, then it is easy to see that the associated cocycle is nothing but the given one c. Now we consider the assumption that $H^1(\Gamma; \mathbf{Z}) = 0$. This condition implies that the 1-cochain d above is uniquely defined so that we can associate a unique section to each cocycle. It is then a routine matter to verify that the above correspondences define a bijection between the set of all sections and that of all cocycles. If we write r_0 for the retraction associated to the fixed section s_0 (or equivalently to the fixed cocycle c_0), then the retraction r corresponding to a general section s (or a general cocycle c) is given by $r(\hat{f}) = r_0(\hat{f}) - d(f)$ where $\hat{f} \in \hat{\Gamma}$ and $f = \pi(\hat{f})$.

Finally the last statement of our proposition can be easily checked by a straightforward inspection. This completes the proof. q.e.d.

Now we finish our digression and go back to our specific central extension $0 \to \mathbf{Z} \to \widehat{\mathcal{M}}_g \to \mathcal{M}_g \to 1$ or rather to a similar one

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widehat{\mathcal{M}}_{g,1} \longrightarrow \mathcal{M}_{g,1} \longrightarrow 1$$

which corresponds to the mapping class group $\mathcal{M}_{g,1}$. It is well known that the mapping class groups $\mathcal{M}_g$ and $\mathcal{M}_{g,1}$ satisfy the assumption of Proposition 3.1 (see [12]). Hence we can apply it to the above central extension. Let $s_0 : \mathcal{M}_{g,1} \to \widehat{\mathcal{M}}_{g,1}$ be the section defined by Atiyah's canonical 2-framing. Then as was mentioned before, Atiyah [4] proved that the associated cocycle is nothing but 3τ, where τ is Meyer's signature cocycle. The retraction $\widehat{\mathcal{M}}_{g,1} \to \mathbf{Z}$ associated to the canonical section s_0 is of course given by thrice the signature defect δ.

Now recall from the previous section that we have our intersection cocycle c which also represents the Euler class of the above central extension. Hence, by Proposition 3.1, we have the associated section

$$s : \mathcal{M}_{g,1} \longrightarrow \widehat{\mathcal{M}}_{g,1}.$$

If we write $c = 3\tau + \delta d$ where $d : \mathcal{M}_{g,1} \to \mathbf{Z}$ is a certain 1-cochain which is uniquely defined, then $s = s_0 + d$. One of the main purposes of the present paper is to give a geometrical meaning to this section s or equivalently to the associated 1-cochain d.

4. Vector fields on surfaces and the mapping class group

Our main concern in this paper is the intersection cocycle $c \in Z^2(\mathcal{M}_{g,1}; \mathbf{Z})$ which represents the first characteristic class e_1. This cocycle is defined once we fix a crossed homomorphism

$$k : \mathcal{M}_{g,1} \to H_1(\Sigma_g; \mathbf{Z})$$

which represents a generator of $H^1(\mathcal{M}_{g,1}; H_1(\Sigma_g; \mathbf{Z})) \cong \mathbf{Z}$. More precisely we have

$$c(\varphi, \psi) = k(\varphi^{-1}) \cdot k(\psi).$$

In our previous papers [20], [21], [24], [26] we showed various ways of constructing crossed homomorphisms $\mathcal{M}_{g,1} \to H_1(\Sigma_g; \mathbf{Z})$ which represent a generator of $H^1(\mathcal{M}_{g,1}; H_1(\Sigma_g; \mathbf{Z}))$. The purpose of this section is to give another crossed homomorphism $\mathcal{M}_{g,1} \to H_1(\Sigma_g; \mathbf{Z})$. This crossed homomorphism was proposed by Furuta [10] and it turned out that it also represents a generator of $H^1(\mathcal{M}_{g,1}; H_1(\Sigma_g; \mathbf{Z}))$, as will be shown below. It is defined in terms of vector fields on surfaces . At present it is the most geometrical one and is fit best to our purpose. Later Trapp [28] defined yet another crossed homomorphism which is given in terms of the concept of winding numbers ([7]) and is closely related to Furuta's one.

Now let $T_1\Sigma_g$ be the unit tangent bundle of Σ_g. Then $\pi_1(T_1\Sigma_g)$ can be naturally considered as a subgroup of $\mathcal{M}_{g,1}$ (see [15]). It was actually proved in [20] that a crossed homomorphism $k : \mathcal{M}_{g,1} \to H_1(\Sigma_g; \mathbf{Z})$ represents a generator of $H^1(\mathcal{M}_{g,1}; H_1(\Sigma_g; \mathbf{Z}))$ if and only if $k(\gamma) = \pm(2-2g)[\gamma]$ for any (and hence all) element $\gamma \in \pi_1(T_1\Sigma_g) \subset \mathcal{M}_{g,1}$ with $[\gamma] \neq 0$ where $[\gamma] \in H_1(\Sigma_g; \mathbf{Z})$ is the homology class of γ projected to the base space Σ_g. Now we define Furuta's crossed homomorphism

$$k : \mathcal{M}_{g,1} \longrightarrow H_1(\Sigma_g; \mathbf{Z}).$$

First choose a non-singular vector field X on $\Sigma_g^0 = \Sigma_g \setminus \operatorname{Int} D^2$, where D^2 is the fixed embedded disc in Σ_g. Also choose a Riemannian metric g_0 on Σ_g. For each element $\varphi \in \mathcal{M}_{g,1}$, select a representative $\varphi \in \operatorname{Diff}_+ \Sigma_g$ (we use the same letter) which is the identity on the disc D^2, and consider the vector field $\varphi_* X$ which is also non-singular. We define a mapping

$$f_\varphi : \Sigma_g^0 \to S^1$$

as follows, where S^1 stands for the set of all *angles* mod 2π. For each point $p \in \Sigma_g^0$, we have two tangent vectors X_p and $(\varphi_* X)_p$ of Σ_g^0 at $p \in \Sigma_g^0$. We set $f_\varphi(p)$ to be $\angle(X_p, (\varphi_* X)_p)$ where the symbol $\angle(X_p, (\varphi_* X)_p)$ denotes the *angle* from X_p to $(\varphi_* X)_p$ which is measured with respect to the fixed orientation and the metric on Σ_g^0 and is counted mod 2π. Let $[f_\varphi] \in H^1(\Sigma_g^0; \mathbf{Z}) = H^1(\Sigma_g; \mathbf{Z})$ be the cohomology class determined by the map f_φ; namely $[f_\varphi] = f_\varphi^*(\iota)$ where $\iota \in H^1(S^1; \mathbf{Z})$ is the generator. Let us write $k(\varphi) \in H_1(\Sigma_g; \mathbf{Z})$ for the Poincaré dual of $[f_\varphi]$. Then we obtain a mapping $k : \mathcal{M}_{g,1} \to H_1(\Sigma_g; \mathbf{Z})$. It is easy to see that the homotopy class of the mapping f_φ does not depend on the choice of the representative of φ in $\mathrm{Diff}_+ \Sigma_g$ nor the choice of the metric g_0 on Σ_g^0 so that the mapping k is well defined depending only on the choice of the non-singular vector field X.

Proposition 4.1. *The mapping $k : \mathcal{M}_{g,1} \to H_1(\Sigma_g; \mathbf{Z})$ defined above is a crossed homomorphism. Moreover its cohomology class $[k]$ is independent of the choice of the non-singular vector field X and is a generator of the infinite cyclic group $H^1(\mathcal{M}_{g,1}; H_1(\Sigma_g; \mathbf{Z}))$.*

Proof. We divide the proof into three steps.

Step I. First we prove that the mapping $k : \mathcal{M}_{g,1} \to H_1(\Sigma_g; \mathbf{Z})$ is a crossed homomorphism. To show this, for any two elements $\varphi, \psi \in \mathcal{M}_{g,1}$ and a point $p \in \Sigma_g^0$, we consider three tangent vectors X_p, $(\varphi_* X)_p$ and $((\varphi\psi)_* X)_p$ at $p \in \Sigma_g^0$. By the definition of the mappings $f_{\varphi\psi}$ and f_φ, we have $f_{\varphi\psi}(p) = \angle(X_p, ((\varphi\psi)_* X)_p)$ and $f_\varphi(p) = \angle(X_p, (\varphi_* X)_p)$. Hence we have

$$f_{\varphi\psi}(p) = f_\varphi(p) + \angle((\varphi_* X)_p, ((\varphi\psi)_* X)_p).$$

Now $\angle((\varphi_* X)_p, ((\varphi\psi)_* X)_p) = \angle(\varphi_*(X_{\varphi^{-1}(p)}), \varphi_*((\psi_* X)_{\varphi^{-1}(p)})$ so that this angle is equal to $\overline{f}_\psi(\varphi^{-1}(p))$ where $\overline{f}_\psi : \Sigma_g^0 \to S^1$ is the mapping defined similarly as f_ψ but with respect to the Riemannian metric $\varphi^*(g_0)$ on Σ_g^0. In particular the two mappings f_ψ and $\overline{f}_\psi$ are homotopic each other. Now we can write

$$f_{\varphi\psi}(p) = f_\varphi(p) + (\overline{f}_\psi \circ \varphi^{-1})(p).$$

If we consider the corresponding cohomology classes, we obtain $[f_{\varphi\psi}] = [f_\varphi] + (\varphi^{-1})^*([f_\psi]) \in H^1(\Sigma_g; \mathbf{Z})$, and taking the Poincaré duals we finally have

$$k(\varphi\psi) = k(\varphi) + \varphi_*(k(\psi)).$$

Hence k is a crossed homomorphism.

Step II. Next we show that the cohomology class $[k]$ is independent of the choice of the non-singular vector field X on Σ_g^0. Let X' be another non-singular vector field on Σ_g^0, and for each element $\varphi \in \mathcal{M}_{g,1}$ we denote $f'_\varphi : \Sigma_g^0 \to S^1$ for the associated map, and also write

$$k' : \mathcal{M}_{g,1} \to H_1(\Sigma_g; \mathbf{Z})$$

for the crossed homomorphism corresponding to X'. Let us write $w : \Sigma_g^0 \to S^1$ for the mapping defined as $w(p) = \angle(X_p, X'_p)$ $(p \in \Sigma_g^0)$, and let $[w] \in H^1(\Sigma_g; \mathbf{Z})$ be the cohomology class determined by w. Also let $\overline{w} : \Sigma_g^0 \to S^1$ be the mapping defined similarly as w but with respect to the Riemannian metric $\varphi^*(g_0)$ on Σ_g^0. Thus $\overline{w}(p) = \angle(\varphi_*(X_p), \varphi_*(X'_p))$. It is easy to see that the two maps w and $\overline{w}$ are homotopic each other so that $[w] = [\overline{w}]$. Now we compute

$$\begin{aligned} f'_\varphi(p) =& \angle(X'_p, (\varphi_* X')_p) \\ =& \angle(X_p, (\varphi_* X)_p) + \angle((\varphi_* X)_p, (\varphi_* X')_p) - \angle(X_p, X'_p) \\ =& f_\varphi(p) + \overline{w}(\varphi^{-1}(p)) - w(p). \end{aligned}$$

Hence we have

$$[f'_\varphi] = [f_\varphi] + (\varphi^{-1})^*([w]) - [w] \in H^1(\Sigma_g; \mathbf{Z}).$$

If we take the Poincaré duals, we obtain

$$k'(\varphi) = k(\varphi) + \varphi_*([w]^*) - [w]^*$$

where $[w]^* \in H_1(\Sigma_g; \mathbf{Z})$ is the Poincaré dual of $[w]$. We can now conclude that the two crossed homomorphisms $k, k' : \mathcal{M}_{g,1} \to H_1(\Sigma_g; \mathbf{Z})$ are cohomologous to each other, because their difference is a coboundary corresponding to $[w]^*$.

Step III. Finally we prove that the cohomology class $[k]$ is a generator of the infinite cyclic group $H^1(\mathcal{M}_{g,1}; H_1(\Sigma_g; \mathbf{Z}))$. To show this, we may assume that $g = 2$ because we know from [20] that the natural inclusion $i : \mathcal{M}_{g,1} \to \mathcal{M}_{g+1,1}$ induces an isomorphism

$$i^* : H^1(\mathcal{M}_{g+1,1}; H_1(\Sigma_{g+1}; \mathbf{Z})) \cong H^1(\mathcal{M}_{g,1}; H_1(\Sigma_g; \mathbf{Z}))$$

for all $g \geq 2$.

FIGURE 1a

FIGURE 1b

Now choose a free generator $\alpha_1, \alpha_2, \beta_1, \beta_2$ of $\pi_1(\Sigma_2^0)$ as depicted in Figure 1a and let x_1, x_2, y_1, y_2 be the corresponding homology classes in $H_1(\Sigma_2; \mathbf{Z})$. Also let m and m' be two simple closed curves on Σ_2^0 as shown in Figure 1b. Set $\gamma = \mu(\mu')^{-1} \in \mathcal{M}_{2,1}$ where μ and μ' are the Dehn twists along m and m' respectively. Then it is easy to see that γ belongs to the subgroup $\pi_1(T_1\Sigma_2) \subset \mathcal{M}_{2,1}$ and also that the homology class of γ projected to the base space Σ_2 is equal to y_2.

Figure 2a

Figure 2b

Now choose a vector field X on Σ_2 as shown in Figure 2a, where we deform the embedded disc D^2, by an isotopy, so that it will contain the two hyperbolic singular points of X in its interior. Thus we obtain a non-singular vector field X on Σ_2^0. Let

$$k_X : \mathcal{M}_{2,1} \longrightarrow H_1(\Sigma_2; \mathbf{Z})$$

be the crossed homomorphism associated to X. We claim that the equality $k_X(\gamma) = 2y_2$ holds. By the above arguments, this will then imply that the crossed homomorphism k_X represents a generator of the infinite cyclic group $H^1(\mathcal{M}_{g,1}; H_1(\Sigma_g; \mathbf{Z}))$ for all $g \geq 2$ and X. More precisely, we have $k_X(\gamma) = (2g-2)[\gamma]$ for all $\gamma \in \pi_1(T_1\Sigma_g) \subset \mathcal{M}_{g,1}$ (notice that the value of $2g-2$ for $g=2$ is equal to 2).

Now we prove $k_X(\gamma) = 2y_2$. We know from the outset that $k_X(\gamma)$ is a multiple of y_2. By the definition of the crossed homomorphism k_X, $k_X(\gamma)$ is the Poincaré dual of the cohomology class $[f_\gamma] = f_\gamma^*(\iota)$ where

$$f_\gamma : \Sigma_2^0 \longrightarrow S^1$$

is the mapping defined by $f_\gamma(p) = \angle(X_p, (\gamma_* X)_p)$. Now we know that $[f_\gamma]$ is a multiple of the Poincaré dual of y_2 which is x_2^*. Here $x_1^*, x_2^*, y_1^*, y_2^*$ are the dual basis of x_1, x_2, y_1, y_2. Hence we have only to count the number of times the point $f_\gamma(p)$ goes around the circle S^1 according as the point p goes around the oriented closed curve ℓ, which is shown in Figure 1b and represents the homology class x_2, once. Since the mapping γ does *not* act on the homology of Σ_g^0, to compute the above number we may replace the mapping f_γ by $f_\gamma \circ \gamma$. Then $(f_\gamma \circ \gamma)(p) = \angle(X_{\gamma(p)}, \gamma_*(X_p))$. Now the vector field X along the oriented closed curve $\gamma(\ell)$ looks like the picture shown in Figure 2b. By a direct inspection, we can now conclude that the above number is equal to 2. This completes the proof. q.e.d.

Since we know that any crossed homomorphism $\mathcal{M}_{g,1} \to H_1(\Sigma_g; \mathbf{Z})$ is trivial on the subgroup $\mathcal{K}_{g,1}$ (see [21], [22]), we have the following immediate corollary to Proposition 4.1.

Corollary 4.2. *Let X be any non-singular vector field on Σ_g^0. Then for any element $\varphi \in \mathcal{K}_{g,1}$, the vector field $\varphi_* X$ is homotopic to X in the sense that there exists a one-parameter family $X_t (t \in \mathbf{R})$ of non-singular vector fields on Σ_g^0 such that $X_0 = X$ and $X_1 = \varphi_* X$.*

5. The Casson invariant as the signature defect of certain framed 3-manifolds

As was explained in the Introduction (see also [22] for details), the influence of the Casson invariant on the structure of the mapping class group is focussed upon an invariant d which defines a homomorphism

$$d : \mathcal{K}_{g,1} \longrightarrow \mathbf{Z}.$$

The value of this invariant d on each generator of $\mathcal{K}_{g,1}$ is very simple. More precisely, let ω be a separating simple closed curve on $\Sigma_g \setminus D^2$ and let h be the genus of the subsurface of Σ_g which is bounded by ω and does not contain the embedded disc D^2. Then $d(D_\omega) = 4h(h-1)$ where D_ω is the Dehn twist along ω.

Now one of our main results is to give a geometrical meaning to this invariant d. To formulate this, for each element $\varphi \in \mathcal{K}_{g,1}$ we consider the mapping torus M_φ which is defined as $\Sigma_g \times I/\sim$ where we identify $(p,0)$ with $(\varphi(p),1)$ $(p \in \Sigma_g)$. We will see below that the 3-manifold M_φ has a canonical framing t_φ on its tangent bundle TM_φ. We can therefore consider the signature defect of the framed 3-manifold (M_φ, t_φ) which we write $\delta(M_\varphi, t_\varphi)$. Namely if Z is a compact oriented 4-manifold with $\partial Z = M_\varphi$, then we have $\delta(M_\varphi, t_\varphi) = \frac{1}{3}p_1(TZ; t_\varphi) - \operatorname{sign} Z$, where $p_1(TZ; t_\varphi)$ denotes the *relative* p_1 of the tangent bundle of Z with respect to the framing t_φ on its boundary.

Theorem 5.1. *Let $\mathcal{K}_{g,1}$ be the subgroup of the mapping class group $\mathcal{M}_{g,1}$ generated by all the Dehn twists along separating simple closed curves on $\Sigma_g \setminus D^2$. Then for each element $\varphi \in \mathcal{K}_{g,1}$, there is a canonical framing t_φ on the mapping torus M_φ such that the equality*

$$d(\varphi) = 3\delta(M_\varphi, t_\varphi)$$

holds, where the right-hand side denotes the signature defect of the framed 3-manifold (M_φ, t_φ).

We now define the framing t_φ on M_φ. For that purpose we first choose a non-singular vector field X on $\Sigma_g^0 = \Sigma_g \setminus \operatorname{Int} D^2$. Since the mapping φ is the identity on D^2, M_φ contains $D^2 \times S^1$ as a natural submanifold. Let us write M'_φ for $M_\varphi \setminus \operatorname{Int} D^2 \times S^1$. Since we are assuming that the mapping class φ belongs to $\mathcal{K}_{g,1}$, the vector field $\varphi_* X$ is homotopic to X, as was proved in the previous section (Corollary 4.2). Hence we can define a non-zero section of $\xi | M'_\varphi$ by making use of a homotopy between X and $\varphi_* X$ where ξ has been used to denote the tangent bundle along the fibres of the surface bundle M_φ. This in turn defines a framing of $\xi | M'_\varphi$ by choosing the second vector of the framing to be the one which is orthogonal to the non-zero section and has the right direction with respect to the orientation on ξ. This induces a framing of TM_φ over M'_φ because $TM_\varphi \cong \xi \oplus \varepsilon$ where ε is the trivial line bundle. Now observe that the restriction of the above framing of ξ to $\partial(D^2 \times S^1)$ is independent of φ. In fact it *rotates* $(2-2g)$-times around the boundary of D^2 and is constant along the direction of S^1. Hence if we glue the tangent bundle of $D^2 \times S^1$ along its boundary according to the above framing and then pull it back to $(D^2/\partial D^2) \times S^1$, then we obtain $(\eta_{2-2g} \hat{\otimes} \varepsilon) \oplus \varepsilon = (\eta_{2-2g} \oplus \varepsilon) \hat{\otimes} \varepsilon$. Here η_{2-2g} is an oriented plane bundle over $D^2/\partial D^2 = S^2$ with the Euler number equal to $2-2g$, ε is the trivial line bundle and the symbol $\hat{\otimes}$ denotes the exterior tensor product. Now the bundle $\eta_{2-2g} \oplus \varepsilon$ over S^2 is trivial and has a unique trivialization up tp homotopy because $\pi_2(\mathrm{SO}(3)) = 0$. Therefore the framing of TM_φ over M'_φ obtained above can be naturally extended to the interior of $D^2 \times S^1$. We thus obtain a global framing t_φ of the 3-manifold M_φ. Unfortunately this framing is not uniquely defined because of the non-vanishing of $H_1(M_\varphi, \mathbf{Z})$. However it *does* hold that the induced 2-framing $2t_\varphi$ is uniquely defined. Moreover it turns out that it does not depend even on the choice of the non-singular vector field X. Hence we can say that the 2-framing $2t_\varphi$ is a canonical one for the 3-manifold M_φ.

Our original proof of Theorem 5.1, which was obtained some years ago, was given by a direct computation of the signature defects. However we think that the best way to prove and also to understand the above results would be to generalize everything from the context of the subgroup $\mathcal{K}_{g,1}$ to that of the whole mapping class group $\mathcal{M}_{g,1}$ as follows. This will allow us to obtain a more conceptual proof of Theorem 5.1 than before. Recall that we have fixed a non-singular vector field X on Σ_g^0. This induces a crossed homomorphism $k_X : \mathcal{M}_{g,1} \longrightarrow H_1(\Sigma_g; \mathbf{Z})$ (see §4) and therefore we have the associated intersection cocycle c_X defined by

$$c_X(\varphi, \psi) = k_X(\varphi^{-1}) \cdot k_X(\psi)(\varphi, \psi \in \mathcal{M}_{g,1})$$

(see §2). According to Proposition 3.1, we have a section

$$s_X : \mathcal{M}_{g,1} \to \widehat{\mathcal{M}}_{g,1}$$

which corresponds to c_X. As in §3, let us write

$$c_X = 3\tau + \delta d_X,$$

where $d_X : \mathcal{M}_{g,1} \to \mathbf{Z}$ is a certain 1-cochain which is uniquely defined. Then $s_X = s_0 + d_X$ where s_0 is Atiyah's canonical section. We have the following result which is a generalization of Theorem 5.1 to the context of the whole mapping class group $\mathcal{M}_{g,1}$.

Theorem 5.2. *Let $\mathcal{M}_{g,1}$ be the mapping class group of Σ_g relative to an embedded disc D^2. Choose a non-singular vector field X on*

$$\Sigma_g^0 = \Sigma_g \setminus \operatorname{Int} D^2.$$

Then for each element $\varphi \in \mathcal{M}_{g,1}$, there is a canonical 2-framing $t_X(\varphi)$ on the mapping torus M_φ, which is uniquely defined, such that the equality

$$d_X(\varphi) = 3\delta(M_\varphi, t_X(\varphi))$$

holds. Here the right-hand side denotes the signature defect of the 2-framed 3-manifold $(M_\varphi, t_X(\varphi))$. *Equivalently, if we define a section* $t_X : \mathcal{M}_{g,1} \to \widehat{\mathcal{M}}_{g,1}$ *by*

$$t_X(\varphi) = (M_\varphi, t_X(\varphi))(\varphi \in \mathcal{M}_{g,1}),$$

then it corresponds to the intersection cocycle c_X*:*

$$c_X(\varphi, \psi) = t_X(\varphi) t_X(\psi) \{t_X(\varphi\psi)\}^{-1}.$$

Here we would like to emphasize that our canonical 2-framing mentioned in the above theorem is *different* from Atiyah's canonical 2-framing which is defined for general 3-manifolds. Now we define our canonical 2-framing $t_X(\varphi)$ on M_φ.

We have the mapping $f_\varphi : \Sigma_g^0 \to S^1$ defined by

$$f_\varphi(p) = \angle(X_p, (\varphi_* X)_p) \quad (p \in \Sigma_g^0),$$

(see §4). We may assume that $\pi \in S^1$ is a regular value of f_φ by perturbing, if necessary, the vector field $\varphi_* X$ slightly (cf. the argument in the proof of Proposition 6.1). Similarly we may assume that the length of the vector fields X and $\varphi_* X$ (after a necessary perturbation) are constantly equal to 1, because it is only the direction of the tangent vectors that is essential in the following arguments and the length is irrelevant. Let us write $K_\varphi \subset \Sigma_g^0$ for the submanifold $f_\varphi^{-1}(\pi)$. We put an orientation on K_φ so that its homology class $[K_\varphi]$ is equal to $k_X(\varphi)$. We also consider K_φ as a submanifold of M_φ by embedding it into $\Sigma_g^0 \times \frac{1}{2} \subset M_\varphi$. Now we define a section σ of the oriented plane bundle ξ over $M'_\varphi = M_\varphi \setminus \operatorname{Int} D^2 \times S^1$ by

$$\sigma(p, t) = (1-t) X_p + t(\varphi_* X)_p \quad (p \in \Sigma_g^0, t \in [0,1]).$$

Then it is easy to see that the image $\operatorname{Im} \sigma$ of this section is transverse to the zero-section Z_0 of ξ and also that $\operatorname{Im} \sigma \cap Z_0 = K_\varphi$. Moreover if we put an orientation on the normal bundle of K_φ which is induced from those on M_φ and K_φ, then it can be checked that the section σ restricted to the normal direction of any point p on K_φ, followed by the projection to the fibre of ξ at p with respect to a trivialization near p, is orientation preserving. We summarize the above argument as the following proposition for later use (compare with Proposition 4.1 of [21]).

Proposition 5.3. *Let $\varphi \in \mathcal{M}_{g,1}$ and let $M_\varphi \to S^1$ be the corresponding oriented Σ_g-bundle. Then the Poincaré dual of the Euler class $\chi(\xi) \in H^2(M_\varphi; \mathbf{Z})$ of the tangent bundle ξ along the fibres of this surface bundle can be represented by the 1-dimensional submanifold*

$$K_\varphi \cup (2-2g)S^1$$

of M_φ where S^1 denotes any "horizontal circle" $q \times S^1 (q \in D^2)$.

Now we go on to define our 2-framing $t_X(\varphi)$ on M_φ. The section σ defines a framing of M'_φ away from the submanifold K_φ. This framing restricted to $\partial(D^2 \times S^1)$ is independent of φ and it has a canonical extension to the interior of $D^2 \times S^1$ as before. Thus we obtain a framing of the tangent bundle of M_φ outside of K_φ and this in turn induces a 2-framing there. Now let us look into the situation near K_φ. Let K be any connected component of K_φ; it is an oriented circle embedded in $\Sigma_g^0 \times \frac{1}{2} \subset M_\varphi$. Let $N(K)$ be a small tubular neighborhood of K in M_φ. Then we can define a bundle trivialization

$$\xi|_{N(K)} \cong N(K) \times \mathbf{R}^2$$

as follows. Recall that X is a non-singular vector field on Σ_g^0. Hence it defines a non-singular vector field over $\Sigma_g^0 \times [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon] \subset M_\varphi$ which is constant along the second factor. Thus we can consider X also as a non-zero section of ξ there. In particular X defines a non-zero section of $\xi|_{N(K)}$. This induces the desired bundle trivialization by sending the vector $X_q (q \in N(K))$ to the first unit vector of $\mathbf{R}^2$. Next let us define an identification $N(K) \cong K \times D^2$ as follows, where D^2 is the unit disc in $\mathbf{R}^2$. For each point $p \in K$, let $D_p \subset N(K)$ denote the normal disc at p so that $N(K) = \cup_{p \in K} D_p$, and let $q_p \in \partial D_p$ be the point such that its second coordinate as a point in $\Sigma_g^0 \times I$ attains the minimum among all the points in D_p. Then the required identification is characterized (up to homotopy) by the condition that each point q_p goes to $(p, 1) \in K \times \partial D^2$. Now if we combine the above two trivializations, we obtain a bundle isomorphism

$$\xi|_{N(K)} \cong (K \times D^2) \times \mathbf{R}^2.$$

By the definition of the section σ, the vector $\sigma(q_p)$ is equal to X_p (up to a positive constant) which is sent to the first unit vector of $\mathbf{R}^2$ under the above isomorphism. Moreover if the point q goes around ∂D_p once in the positive direction starting from the point q_p, then the corresponding vector $\sigma(q)$, projected to the fibre of ξ at p, goes around the origin once in the positive direction starting from the vector X_p. Hence, by perturbing if necessary the section σ appropriately, we can conclude that the section $\bar{\sigma}$ of the trivial bundle $(K \times D^2) \times \mathbf{R}^2$, which is induced from σ via the above bundle isomorphism, is the "identity". Namely we have $\bar{\sigma}(p, v) = v$ $(p \in K, v \in D^2 \subset \mathbf{R}^2)$. Hence if we glue the bundle ξ over $N(K) = K \times D^2$ along its boundary according to the framing there induced from the section σ and pull it back to $K \times (D^2/\partial D^2)$, then we obtain $\varepsilon \hat{\otimes} \eta_1$. Here ε is the trivial line bundle over K, and η_1 denotes an oriented plane bundle over $D^2/\partial D^2 = S^2$ with the Euler number exactly equal to 1. Since $\pi_2(\mathrm{Spin}(4)) = 0$, $2\eta_1$ has a unique trivialization as a Spin(4)-bundle over S^2. Hence we can conclude that the above 2-framing of M_φ, which was defined outside of K_φ, can be uniquely extended to the whole of M_φ as a global 2-framing. This is the definition of our canonical 2-framing $t_X(\varphi)$. We will see later that this 2-framing is well defined independent of the various choices made (see Proposition 6.1).

6. Proofs of Theorem 5.1 and Theorem 5.2

In this section we prove our main results (Theorem 5.1 and Theorem 5.2). First we show

Proof of Theorem 5.1 assuming Theorem 5.2. Let X be any non-singular vector field on Σ_g^0. Then we know that for any element $\varphi \in \mathcal{K}_{g,1}$, the vector field $\varphi_* X$ is homotopic to X or equivalently we have $k_X(\varphi) = 0$ (see Corollary 4.2). Hence, by the definitions, the 2-framing $t_X(\varphi)$ of the mapping torus M_φ is equal to twice the usual framing t_φ. It follows that $\delta(M_\varphi, t_\varphi) = \delta(M_\varphi, t_X(\varphi))$. On the other hand, we know also that the value $d_X(\varphi)$ is independent of the choice of X and is equal to $d(\varphi)$ (see [22]). It is now clear that Theorem 5.1 follows from Theorem 5.2. In particular we have proved that the 2-framing $2t_\varphi$ does not depend on the choice of the non-singular vector field X. q.e.d.

Next we prove Theorem 5.2. For that, we first generalize the definition of the 2-framing $t_X(\varphi)$ given after Proposition 5.3 as follows. Thereby we also prove that this 2-framing is uniquely defined.

Proposition 6.1. *Let $\varphi \in \mathcal{M}_{g,1}$ and let M_φ be the corresponding mapping torus. Let K_φ be any oriented 1-dimensional submanifold of M_φ embedded in the locus $\Sigma_g^0 \times \frac{1}{2} \subset M_\varphi$ such that its homology class* *in Σ_g^0 is equal to $k_X(\varphi)$. Then there exists a section σ of the oriented plane bundle ξ over M'_φ such that:*

(i) its value at any point $p \times 0 \in \Sigma_g^0 \times 0$ is equal to X_p,

(ii) it is transverse to the zero-section Z_0 of ξ, and $\operatorname{Im}\sigma \cap Z_0 = K_\varphi$.

Moreover the corresponding 2-framing, which is defined in a similar way as $t_X(\varphi)$, is homotopic to $t_X(\varphi)$. It follows in particular that the homotopy class of this 2-framing depends only on X and φ and not on the choice of the submanifold K_φ.

Proof. As before, we consider the mapping $f_\varphi : \Sigma_g^0 \to S^1$ defined by $f_\varphi(p) = \angle(X_p, (\varphi_* X)_p)$ $(p \in \Sigma_g^0)$, and assume that $\pi \in S^1$ is a regular value of f_φ. We put $K_\varphi^0 = f_\varphi^{-1}(\pi)$, and consider it as an oriented submanifold of M_φ embedded in $\Sigma_g^0 \times \frac{1}{2} \subset M_\varphi$. Our 2-framing $t_X(\varphi)$ was defined in terms of this picture. Now we compare it with the more general situation described in Proposition 6.1. It is easy to see that we can construct a smooth mapping $f : \Sigma_g^0 \to S^1$ such that (i) f sends any point on $\partial\Sigma_g^0$ to $0 \in S^1$ and (ii) $\pi \in S^1$ is a regular value of f and $f^{-1}(\pi)$ coincides with the given 1-dimensional submanifold $K_\varphi \subset \Sigma_g^0 \times \frac{1}{2}$. Since the homology class of K_φ is equal to $k_X(\varphi)$, by the assumption, the mapping f is homotopic to f_φ relative to $\partial\Sigma_g^0$. Now let X' be a non-singular vector field on Σ_g^0 defined by the conditions $\angle(X_p, X'_p) = f(p)$ and $\|X'_p\| = \|X_p\|(p \in \Sigma_g^0)$. Since f is homotopic to f_φ, it is easy to see that X' is homotopic to $\varphi_* X$. Now we define a section σ of ξ over M'_φ by

$$\sigma(p,t) = \begin{cases} X_p & t \in [0, \frac{1}{4}], \\ (\frac{3}{2} - 2t)X_p + (2t - \frac{1}{2})X'_p & t \in [\frac{1}{4}, \frac{3}{4}], \\ X_p^t & t \in [\frac{3}{4}, 1], \end{cases}$$

where $p \in \Sigma_g^0$, and $X^t (t \in [\frac{3}{4}, 1])$ is a homotopy of non-singular vector fields such that $X^{\frac{3}{4}} = X'$ and $X^1 = \varphi_* X$. Then it is easy to see that σ is transverse to the zero-section Z_0 of ξ and $\operatorname{Im}\sigma \cap Z_0 = K_\varphi$. This proves (i).

Next we prove that the 2-framing associated to the above picture is homotopic to the original $t_X(\varphi)$. Since the two mappings $f_\varphi, f : \Sigma_g^0 \to S^1$ are homotopic to each other, there is a homotopy $F : \Sigma_g^0 \times I \to S^1$ such that $F(p,0) = f_\varphi(p)$ and $F(p,1) = f(p)$. We may assume that $\pi \in S^1$ is a regular value of F, and put $\widetilde{K} = F^{-1}(\pi)$. Then $\widetilde{K}$ is a submanifold of $\Sigma_g^0 \times I$ of codimension one such that it meets the boundary transversely, and $\widetilde{K} \cap \Sigma_g^0 \times 0 = K_\varphi^0$, $\widetilde{K} \cap \Sigma_g^0 \times 1 = K_\varphi$. We consider $\widetilde{K}$ as a submanifold of $M_\varphi \times I$ by embedding it into the locus $\Sigma_g^0 \times \frac{1}{2} \times I \subset M_\varphi \times I$, and let $N(\widetilde{K})$ denote a small tubular neighborhood of $\widetilde{K}$ in $M_\varphi \times I$. Now we consider the oriented plane bundle $\widetilde{\xi} = \xi \hat{\otimes} \varepsilon$ over $M_\varphi \times I$ where ε denotes the trivial line bundle over I. We define a bundle trivialization

$$\widetilde{\xi}|_{\widetilde{K}} \cong \widetilde{K} \times \mathbf{R}^2$$

as follows. The non-singular vector field X on Σ_g^0 defines the one on a neighborhood of $\Sigma_g^0 \times \frac{1}{2} \times I$, which is constant along the second as well as the third factor. In particular we can consider X as a non-zero section of $\widetilde{\xi}|_{N(\widetilde{K})}$. Then the required trivialization is defined by sending each vector $X_{\tilde{q}} (\tilde{q} \in N(\widetilde{K}))$ to the first unit vector of $\mathbf{R}^2$.

Now it can be easily shown that the section of $\widetilde{\xi}$ given over $M'_\varphi \times \{0,1\}$ can be extended to a section $\widetilde{\sigma}$ which is defined over $M'_\varphi \times I$ such that it is transverse to the zero-section $\widetilde{Z}_0$ of $\widetilde{\xi}$ and $\operatorname{Im} \widetilde{\sigma} \cap \widetilde{Z}_0 = \widetilde{K}$. This section $\widetilde{\sigma}$ induces a trivialization $N(\widetilde{K}) = \widetilde{K} \times D^2$ as follows. For each point $\tilde{p} \in \widetilde{K}$, let $D_{\tilde{p}}$ be the normal disc at $\tilde{p}$ so that we have $N(\widetilde{K}) = \cup_{\tilde{p} \in \widetilde{K}} D_{\tilde{p}}$. By the condition that $\widetilde{\sigma}$ is transverse to the zero-section, we see that there is (essentially) a unique point $\tilde{q}_{\tilde{p}} \in \partial D_{\tilde{p}}$ such that the vector $\widetilde{\sigma}(\tilde{q}_{\tilde{p}})$ is equal to $X_{\tilde{q}_{\tilde{p}}}$ up to a positive scalar. Then the required trivialization is defined by sending the point $\tilde{q}_{\tilde{p}}$ to $(\tilde{p}, 1)$. It is easy to see that the restriction of the above trivialization to the boundary coincides with those of $N(K_\varphi^0)$ and $N(K_\varphi)$. Thus we obtain a bundle isomorphism

$$\widetilde{\xi}|_{\widetilde{K}} \cong (\widetilde{K} \times D^2) \times \mathbf{R}^2,$$

which coincides with the previously given one on the boundary. Then by exactly the same reason as before, the section $\bar{\sigma}$ of the trivial bundle $(\widetilde{K} \times D^2) \times \mathbf{R}^2$ which is induced from $\widetilde{\sigma}$ via the above isomorphism is given by the "identity", namely,

$$\bar{\sigma}(\tilde{p}, v) = v (\tilde{p} \in \widetilde{K}, v \in D^2 \subset \mathbf{R}^2).$$

If we glue the bundle $\widetilde{\xi}$ over $N(\widetilde{K})$ along its boundary according to the framing there induced from the non-zero section $\widetilde{\sigma}$ and pull it back to $\widetilde{K} \times (D^2/\partial D^2)$, then we obtain $\varepsilon \hat{\otimes} \eta_1$ where ε is the trivial line bundle over $\widetilde{K}$. Hence the 2-framing on $M'_\varphi \times \{0,1\}$, which is a usual framing of the Spin(6)-bundle $2TM_\varphi = 2(\xi \oplus \varepsilon)$, can be extended to a framing of $2TM_\varphi\hat{\hat{\otimes}}\varepsilon$ over $M'_\varphi \times I$. On the other hand, the framings of $\xi \oplus \varepsilon$ over each embedded solid torus $D^2 \times S^1$ in $M_\varphi \times 0$ and $M_\varphi \times 1$ are the same so that they can be extended constantly over $D^2 \times S^1 \times I \subset M_\varphi \times I$. Thus we have shown that the 2-framing on $M_\varphi \times \{0,1\}$ can be extended to a global framing of $2TM_\varphi\hat{\hat{\otimes}}\varepsilon$ over the whole of $M_\varphi \times I$. This completes the proof of (ii) and hence that of Proposition. q.e.d.

Now we begin the proof of Theorem 5.2.

Proof of Theorem 5.2. The situation is as follows. We fix a non-singular vector field X on Σ_g^0. We then have the associated intersection cocycle c_X given by

$$c_X(\varphi,\psi) = k_X(\varphi^{-1}) \cdot k_X(\psi)(\varphi,\psi \in \mathcal{M}_{g,1}),$$

where $k_X : \mathcal{M}_{g,1} \to H_1(\Sigma_g;\mathbf{Z})$ is the crossed homomorphism corresponding to X. Let us define a section $t_X : \mathcal{M}_{g,1} \to \widehat{\mathcal{M}}_{g,1}$ by

$$t_X(\varphi) = (M_\varphi, t_X(\varphi))(\varphi \in \mathcal{M}_{g,1}),$$

where $t_X(\varphi)$ is the 2-framing on M_φ defined in §5. Then our task is to prove the identity

$$c_X(\varphi,\psi) = t_X(\varphi)t_X(\psi)\{t_X(\varphi\psi)\}^{-1}$$

for all $\varphi,\psi \in \mathcal{M}_{g,1}$. Now we consider the Σ_g-bundle

$$M_{(\varphi,\psi)} \longrightarrow P,$$

where P is a pair of pants (or the 2-sphere with three holes) such that its boundary is a disjoint union of M_φ, M_ψ and $-M_{\varphi\psi}$. Now the value $t_X(\varphi)t_X(\psi)\{t_X(\varphi\psi)\}^{-1}$ is equal to the "difference degree" of the two 2-framings $t_X(\varphi)t_X(\psi)$ and $t_X(\varphi\psi)$ on $M_{\varphi\psi}$. In other words, it is equal to

$$-\frac{1}{2}p_1\Big(2T(M_{\varphi\psi} \times I); -t_X(\varphi)t_X(\psi) + t_X(\varphi\psi)\Big),$$

which is minus half of the relative p_1 of twice the tangent bundle of the 4-manifold $M_{\varphi\psi} \times I$ with respect to the 2-framing $-t_X(\varphi)t_X(\psi)$ on $-M_{\varphi\psi} \times 0$ and $t_X(\varphi\psi)$ on $M_{\varphi\psi} \times 1$. On the other hand, by the definition of the multiplication in the group $\widehat{\mathcal{M}}_{g,1}$, the relative p_1 of the 4-manifold $M_{(\varphi,\psi)}$ with respect to the 2-framing $t_X(\varphi) + t_X(\psi) - t_X(\varphi)t_X(\psi)$ on its boundary vanishes. Hence we can conclude that

$$t_X(\varphi)t_X(\psi)\{t_X(\varphi\psi)\}^{-1} = \frac{1}{2}p_1\Big(2TM_{(\varphi,\psi)}; t_X(\varphi) + t_X(\psi) - t_X(\varphi\psi)\Big).$$

Thus we have only to prove the following equality

$$\frac{1}{2}p_1\Big(2TM_{(\varphi,\psi)}; t_X(\varphi)+t_X(\psi)-t_X(\varphi\psi)\Big) = k_X(\varphi^{-1})\cdot k_X(\psi).$$

Namely our task is to prove that the half of the relative p_1 of the 4-manifold $M_{(\varphi,\psi)}$ with respect to our canonical 2-framings on each connected component of its boundary is given by the intersection number $k_X(\varphi^{-1})\cdot k_X(\psi) = -k_X(\varphi)\cdot\varphi_*(k_X(\psi))$.

Now instead of the Σ_g-bundle $M_{(\varphi,\psi)}$ over P, we consider the following model for it. Namely we take the product bundle $\Sigma_g\times\triangle$ over the standard 2-simplex $\triangle$ spanned by the three vertices $\{0,1,2\}$, and then identify the point $(p,0)$ with $(\varphi(p),1)$ and $(\varphi\psi(p),2)$ for all $p\in\Sigma_g$. The resulting Σ_g-bundle is essentially the same as $M_{(\varphi,\psi)}$. We have the non-singular vector field X on $\Sigma_g^0\times 0$ and it corresponds to φ_*X (resp. $(\varphi\psi)_*X$) on $\Sigma_g^0\times 1$ (resp. $\Sigma_g^0\times 2$) under the above identification. Now choose any oriented 1-dimensional submanifold K_φ, K_ψ and $K_{\varphi\psi}$ of $\operatorname{Int}\Sigma_g^0$ whose homology classes are $k_X(\varphi)$, $k_X(\psi)$ and $k_X(\varphi\psi)$ respectively. We embed the three manifolds K_φ, $\varphi(K_\psi)$, and $K_{\varphi\psi}$ into the locus $\Sigma_g^0\times m_{01}$, $\Sigma_g^0\times m_{12}$ and $\Sigma_g^0\times m_{02}$ respectively, where $m_{ij}\in\triangle$ is the midpoint of the 1-simplex ℓ_{ij} spanned by the two vertices $\{i,j\}$. We have the identity $k_X(\varphi\psi)=k_X(\varphi)+\varphi_*(k_X(\psi))$ so that there exists an oriented properly embedded submanifold W of $\Sigma_g\times\triangle$ of codimension two such that $\partial W = K_\varphi\cup\varphi(K_\psi)\cup -K_{\varphi\psi}$ (see Figure 3).

FIGURE 3

Now let ξ be the tangent bundle along the fibre of the trivial Σ_g-bundle $\Sigma_g \times \triangle$. The boundary of this trivial bundle can be written as

$$\partial(\Sigma_g \times \triangle) = \Sigma_g \times \ell_{01} \cup \Sigma_g \times \ell_{12} \cup -\Sigma_g \times \ell_{02}$$

and each of the product bundle $\Sigma_g \times \ell_{ij}$ on the right-hand side will serve as a model for the Σ_g-bundles M_φ, $\varphi(M_\psi)$ and $M_{\varphi\psi}$ respectively. Here the meaning of the second bundle $\varphi(M_\psi)$ is the following. Namely we transfer everything from the bundle M_ψ to $\Sigma_g \times \ell_{12}$ by the map φ (see Figure 3). By virtue of Proposition 6.1, We have the section σ of ξ over $\Sigma_g^0 \times \partial\triangle$, which is transverse to the zero-section Z_0 such that $\operatorname{Im}\sigma \cap Z_0 = K_\varphi \cup \varphi(K_\psi) \cup -K_{\varphi\psi}$. We also have the associated 2-framing $t_X(\varphi) + \varphi(t_X(\psi)) - t_X(\varphi\psi)$ on the boundary. Clearly

$$\begin{aligned} p_1\Big(2TM_{(\varphi,\psi)}; t_X(\varphi) + t_X(\psi) - t_X(\varphi\psi)\Big) \\ = p_1\Big(2T(\Sigma_g \times \triangle); t_X(\varphi) + \varphi(t_X(\psi)) - t_X(\varphi\psi)\Big). \end{aligned}$$

In what follows, we will compute the right-hand side of the above equality. Now it is easy to see that the section σ of ξ defined on $\Sigma_g^0 \times \partial\triangle$ can be extended to a global section $\widetilde{\sigma}$ over the whole of $\Sigma_g^0 \times \triangle$ such that it is transverse to the zero-section $\widetilde{Z}_0$ and $\operatorname{Im}\widetilde{\sigma} \cap \widetilde{Z}_0 = W$. Let $N(W)$ be a small tubular neighborhood of W. Then, as before, we can define a trivialization

$$\xi|_{N(W)} \cong N(W) \times \mathbf{R}^2$$

as follows. We consider X as a non-singular vector field over a neighborhood of W in $\Sigma_g \times \triangle$, which is constant along the factor $\triangle$. In particular X defines a non-zero section of $\xi|_{N(W)}$. Then the required trivialization is given by sending each vector $X_q (q \in N(W))$ to the first unit vector of $\mathbf{R}^2$. Next for each point $p \in W$ let D_p be the normal disc at p so that $N(W) = \cup_{p\in W} D_p$. Since the section $\widetilde{\sigma}$ is transverse to the zero-section, there exists a unique point $q_p \in D_p$ such that the vector $\widetilde{\sigma}(q_p)$ is equal X_{q_p} up to a positive constant. Then we can define a trivialization $N(W) \cong W \times D^2$ by sending each point q_p to $(p, 1)$. Thus we have a bundle isomorphism

$$\xi|_{N(W)} \cong (W \times D^2) \times \mathbf{R}^2,$$

and the section $\bar{\sigma}$ of the trivial bundle $(W \times D^2) \times \mathbf{R}^2$ induced from $\widetilde{\sigma}$ via the above isomorphism is given by the "identity", namely $\bar{\sigma}(p, v) = v$ $p \in W, v \in D^2 \subset \mathbf{R}^2$. Now the section $\widetilde{\sigma}$ defines a framing of ξ over

$\Sigma_g^0 \times \triangle \setminus \operatorname{Int} N(W)$. If we glue the bundle $\xi|_{N(W)}$ along its boundary according to the above framing and pull it back to $W \times D^2/\partial D^2$, then we obtain $\varepsilon\hat{\otimes}\eta_1$. By trivializing $2\eta_1$, we get a framing of 2ξ over $\Sigma_g^0 \times \triangle$. Clearly the induced framing of 2ξ on $\Sigma_g^0 \times \ell_{01}$ and $\Sigma_g^0 \times \ell_{02}$ extends to the 2-framings $t_X(\varphi)$ and $t_X(\varphi\psi)$ respectively. However the induced 2-framing on $\Sigma_g^0 \times \ell_{12}$, which we denote $t'_X(\psi)$, is different from the original 2-framing $\varphi(t_X(\psi))$ in general. Now it is easy to see that

$$p_1\Big(2T(\Sigma_g \times \triangle); t_X(\varphi) + t'_X(\psi) - t_X(\varphi\psi)\Big) = 0,$$

because the above framing of 2ξ is defined over the whole of $\Sigma_g \times \triangle$. Hence we can conclude that

$$p_1\Big(2T(\Sigma_g \times \triangle); t_X(\varphi) + \varphi(t_X(\psi)) - t_X(\varphi\psi)\Big) = 2\delta(\varphi(t_X(\psi)), t'_X(\psi)),$$

where the right-hand side denotes twice the "difference degree" (= half of the *relative* p_1) of the two 2-framings $\varphi(t_X(\psi))$, $t'_X(\psi)$ of $\Sigma_g \times \ell_{12}$.

Claim. We have the equality

$$\delta(\varphi(t_X(\psi)), t'_X(\psi)) = -k_X(\varphi) \cdot \varphi_*(k_X(\psi)).$$

The above claim will surely complete the proof of Theorem 5.2. We first remark that it is enough to prove that the difference degree is *proportional* to the intersection number $k_X(\varphi) \cdot \varphi_*(k_X(\psi))$. For if it is the case, then it follows that the 2-cocycle

$$\mathcal{M}_{g,1} \ni \varphi, \psi \mapsto t_X(\varphi) t_X(\psi) \{t_X(\varphi\psi)\}^{-1}$$

is proportional to the intersection cocycle $c_X(\varphi, \psi)$. But we know that the above two cocycles are cohomologous to each other and both of them represent the same class $e_1 \in H^2(\mathcal{M}_{g,1}; \mathbf{Z})$ which is nontrivial. Hence they must be equal.

Now we prove the above weakened form of our Claim. First observe that the two 2-framings $\varphi(t_X(\psi))$, $t'_X(\psi)$ are the same outside of a neighborhood of $\varphi(K_\psi)$ so that their difference is concentrated on $N(\varphi(K_\psi))$. To analyse this difference, let us write down explicit formulae for the framings of 2ξ over $N(\varphi(K_\psi))$ which are induced from the above two 2-framings. First we consider the one induced from the 2-framing $t'_X(\psi)$. Recall that this framing was defined in terms of the bundle isomorphism $\xi|_{N(W)} \cong (W \times D^2) \times \mathbf{R}^2$, which is induced from the non-singular vector field X near W. We have the associated bundle projection $\xi|N(W) \to D^2 \times \mathbf{R}^2$ followed by the mapping

$$\begin{array}{ccc} D^2 \times \mathbf{R}^2 & \longrightarrow & E(\eta_1) \\ \downarrow & & \downarrow \\ D^2 & \longrightarrow & D^2/\partial D^2 = S^2 \end{array}$$

which is induced from the framing on ∂D^2 given by the non-zero section $\partial D^2 \ni v \mapsto v \in \mathbf{R}^2$ ($v \in \partial D^2 \subset \mathbf{R}^2$) there. Now let $R_\theta \in \mathrm{SO}(2)$ be the rotation matrix corresponding to the angle $\theta \in S^1$. Then the framing on ∂D^2 is given by $\partial D^2 \times \mathbf{R}^2 \ni (v, \zeta) \mapsto R_\theta^{-1}\zeta \in \mathbf{R}^2$ where $v = e^{i\theta} \in \partial D^2$. Now let $T_\theta = R_\theta \oplus R_\theta \in \mathrm{SO}(2) \times \mathrm{SO}(2) \subset \mathrm{SO}(4)$. It is easy to see that the mapping $S^1 \ni \theta \mapsto T_\theta \in \mathrm{SO}(4)$ is null homotopic. In fact, if we set

$$T_{\theta,r} = \begin{pmatrix} r\cos\theta & -r\sin\theta & \sqrt{1-r^2} & 0 \\ r\sin\theta & r\cos\theta & 0 & -\sqrt{1-r^2} \\ -\sqrt{1-r^2} & 0 & r\cos\theta & -r\sin\theta \\ 0 & \sqrt{1-r^2} & r\sin\theta & r\cos\theta \end{pmatrix} \in \mathrm{SO}(4),$$

then we have $T_{\theta,1} = T_\theta$ and the correspondence

$$D^2 \ni v = re^{i\theta} \mapsto T_{\theta,r} \in \mathrm{SO}(4)$$

defines an extension of the above mapping to D^2. In terms of these matrices, our framing of $2\eta_1$ is given, at the level of the trivial bundle $D^2 \times (\mathbf{R}^2 \oplus \mathbf{R}^2)$, by

$$D^2 \times (\mathbf{R}^2 \oplus \mathbf{R}^2) \ni (v = re^{i\theta}, \zeta) \mapsto T_{\theta,r}^{-1}\zeta \in \mathbf{R}^4.$$

Now let K be any connected component of $\varphi(K_\psi)$. Then, by the above argument, the framing of 2ξ over $N(K)$ corresponding to the 2-framing $t'_X(\psi)$ is given by

$$2\xi|_{N(K)} \cong (K \times D^2) \times (\mathbf{R}^2 \oplus \mathbf{R}^2) \ni (p, v, \zeta) \mapsto T_{\theta,r}^{-1}\zeta \in \mathbf{R}^4,$$

where $p \in K$, $v = re^{i\theta} \in D^2$ and $\zeta \in \mathbf{R}^2 \oplus \mathbf{R}^2 = \mathbf{R}^4$ as before.

Next we consider the framing of 2ξ over $N(K)$ corresponding to the 2-framing $\varphi(t_X(\psi))$. The difference of this framing from the former one comes entirely from that of the choice of the bundle isomorphism $\xi|_{N(K)} \cong (K \times D^2) \times \mathbf{R}^2$. This time, instead of X we have to use the non-singular vector field $\varphi_* X$, which is now considered as a vector field defined near $N(K)$. For each point $p \in K$, let $D_p \subset N(K)$ be the normal disc at p. We have the identification $\xi|_{D_p} \cong D^2 \times \mathbf{R}^2$ which is defined by the vector field X. Then the vector field $\varphi_* X$ on D_p is represented by a section $\sigma_p : D^2 \to \mathbf{R}^2$ of the trivial bundle $D^2 \times \mathbf{R}^2$ via the above identification. Now by perturbing the vector field $\varphi_* X$ appropriately, we may assume that the above section is a constant function for all p. Namely if we set $u(p) = \angle(X_p, (\varphi_* X)_p) \in S^1$, then we have $\sigma_p(v) = u(p)$. Here we understand $u(p)$ as a vector in $\mathbf{R}^2$ of length 1. It follows that the bundle trivialization of $\xi|_{D_p}$ with respect to the vector field $\varphi_* X$ is given by the composition

$$\xi|_{D_p} \cong D^2 \times \mathbf{R}^2 \to D^2 \times \mathbf{R}^2,$$

where the first isomorphism is the one with respect to the vector field X, and the second bundle map is given by

$$D^2 \times \mathbf{R}^2 \ni (v, \zeta) \mapsto (R_{u(p)}^{-1} v, R_{u(p)}^{-1} \zeta) \in D^2 \times \mathbf{R}^2.$$

Hence the bundle trivialization of $\xi|_{N(K)}$ with respect to the vector field $\varphi_* X$ is given by the composition

$$\xi|_{N(K)} \cong (K \times D^2) \times \mathbf{R}^2 \to (K \times D^2) \times \mathbf{R}^2,$$

where the first isomorphism is the one corresponding to X, and the second mapping is given by

$$(K \times D^2) \times \mathbf{R}^2 \ni (p, v, \zeta) \mapsto (p, R_{u(p)}^{-1} v, R_{u(p)}^{-1} \zeta) \in (K \times D^2) \times \mathbf{R}^2.$$

Observe here that if we write $v = re^{i\theta}$, then $R_{u(p)}^{-1} v = re^{i(\theta - u(p))}$. We can now obtain the following formula for the framing of 2ξ over $N(K)$ corresponding to the 2-framing $\varphi(t_X(\psi))$:

$$2\xi|_{N(K)} \cong (K \times D^2) \times (\mathbf{R}^2 \oplus \mathbf{R}^2) \ni (p, v, \zeta) \mapsto T_{\theta - u(p), r}^{-1} T_{u(p)}^{-1} \zeta \in \mathbf{R}^4,$$

where $p \in K$, $v = re^{i\theta} \in D^2$, and $\zeta \in \mathbf{R}^4$.

If we compare the above explicit formulae for the two framings of $2\xi|_{N(K)}$ corresponding to the 2-framings $\varphi(t_X(\psi))$ and $t'_X(\psi)$, we can conclude that their difference can be described by the mapping

$$K \times D^2 \ni (p, v) \mapsto T_{\theta, r}^{-1} T_{u(p)} T_{\theta - u(p), r} \in \mathrm{SO}(4),$$

which factors through a mapping

$$h : K \times S^2 \to \mathrm{SO}(4).$$

More precisely, if K_i runs through all connected components of $\varphi(K_\psi)$, and $h_i : K_i \times S^2 \to \mathrm{SO}(4)$ denotes the corresponding mapping, then

$$\delta(\varphi(t_X(\psi)), t'_X(\psi)) = \frac{1}{2} \sum_i < [K_i \times S^2], h_i^*(CS) > .$$

Here $[K_i \times S^2]$ is the fundamental homology class of $K_i \times S^2$, and $CS \in H^3(\mathrm{SO}(4); \mathbf{Z})$ denotes the Chern-Simons class corresponding to the first Pontrjagin class p_1. Now let us define a mapping

$$h_0 : S^1 \times S^2 \to \mathrm{SO}(4)$$

by the correspondence

$$S^1 \times D^2 \ni (u, v) \mapsto T_{\theta,r}^{-1} T_u T_{\theta-u,r} \in \mathrm{SO}(4),$$

and set $c_0 =< [S^1 \times S^2], h_0^*(CS) >$. If we define $h'_i : K \times S^2 \to S^1 \times S^2$ by $h'_i(p, v) = (u(p), v)$, then clearly we have $h_i = h_0 \circ h'_i$. Hence if we write $\deg h'_i$ for the degree of the mapping h'_i, then

$$\delta(\varphi(t_X(\psi)), t'_X(\psi)) = \frac{1}{2} c_0 \sum_i \deg h'_i.$$

Subclaim. We have the equality

$$\sum_i \deg h'_i = -k_X(\varphi) \cdot \varphi_*(k_X(\psi)).$$

Proof of Subclaim. Clearly we have $\deg h'_i = \deg u_i$ where, recall that, for each connected component K_i of $\varphi(K_\psi)$, the mapping $u_i : K_i \to S^1$ was defined by $u_i(p) = \angle(X_p, (\varphi_* X)_p)$. On the other hand, we have the mapping $f_\varphi : \Sigma_g^0 \to S^1$ defined by $f_\varphi(p) = \angle(X_p, (\varphi_* X)_p)$ $(p \in \Sigma_g^0)$ so that u_i is nothing but the restriction of f_φ to $K_i \subset \Sigma_g^0$. Since $f_\varphi^*(\iota) \in H^2(\Sigma_g; \mathbf{Z})$ is the Poincaré dual of $k_X(\varphi)$, we have the equality $< u, f_\varphi^*(\iota) > = u \cdot k_X(\varphi)$ which holds for any homology class $u \in H_1(\Sigma_g; \mathbf{Z})$. Now we compute

$$\begin{aligned} \sum_i \deg h'_i &= \sum_i \deg u_i \\ &= < [\varphi(K_\psi)], f_\varphi^*(\iota) > \\ &= -\, k_X(\varphi) \cdot \varphi_*(k_X(\psi)). \end{aligned}$$

This completes the proof of Subclaim.

Thus we have the equality

$$\delta(\varphi(t_X(\psi)), t'_X(\psi)) = -\frac{1}{2}c_0 k_X(\varphi) \cdot \varphi_*(k_X(\psi)),$$

and this in turn completes the proof of Claim and hence that of Theorem 5.2. q.e.d.

Remark 6.1. As an immediate consequence of the above proof, we can conclude that the value of c_0 is equal to 2. It is possible to determine this value by an explicit computation. But our conceptual proof as above enables us to avoid this rather troublesome work.

7. The secondary characteristic classes of surface bundles

Up to the previous section, we have given an interpretation of the Casson invariant in terms of Hirzebruch's signature defect of certain framed 3-manifolds. More precisely, we identified the value of the homomorphism

$$d : \mathcal{K}_{g,1} \longrightarrow \mathbf{Z}$$

which serves as the core of the Casson invariant from our point of view, with the signature defect of the corresponding mapping torus with respect to some canonical framing of its tangent bundle (Theorem 5.1). In this section we generalize the above procedure but, so to speak, in the opposite direction we define *secondary* characteristic classes of surface bundles. Namely we make use of the concept of the signature defect to obtain a series of certain cohomology classes

$$d_i \in H^{4i-3}(\mathcal{K}_{g,1}; \mathbf{Q})^{\mathcal{M}_{g,1}} \quad (i = 1, 2, \cdots)$$

of which the first one d_1 is nothing but the above invariant $d : \mathcal{K}_{g,1} \to \mathbf{Z}$ (up to the factor $\frac{1}{3}$) which is known to be $\mathcal{M}_{g,1}$-invariant (see [22], [23]).

To define our secondary classes, we use the generalized form of Thom's bordism theory given in [1], [8]. Thus for any CW complex K, let $\Omega_*(K) = \sum_i \Omega_i(K)$ denote the oriented bordism group of K. It has a natural structure of a graded module over the Thom bordism ring Ω_*. Over the rationals, there exists a natural isomorphism

$$\Omega_i(K) \otimes \mathbf{Q} \cong \sum_{p+q=i} H_p(K; \mathbf{Q}) \otimes \Omega_q$$

(see [8]). The following is an immediate corollary to this statement.

Proposition 7.1. *Let K be a CW complex and let*

$$f : \Omega_i(K) \longrightarrow \mathbf{Q}$$

be a homomorphism. Assume that f vanishes on the submodule

$$\sum_{p+q=i, q>0} H_p(K;\mathbf{Q}) \otimes \Omega_q \subset \Omega_i(K) \otimes \mathbf{Q}.$$

Then f can be considered as an element of $H^i(K;\mathbf{Q})$.

Let us write $\Omega_i(\mathcal{K}_{g,1})$ for the bordism group of $B\mathcal{K}_{g,1}$ which is the classifying space for the group $\mathcal{K}_{g,1}$. Hereafter we are going to define a homomorphism

$$\delta_i : \Omega_{4i-3}(\mathcal{K}_{g,1}) \longrightarrow \mathbf{Q}.$$

Any element of $\Omega_i(\mathcal{K}_{g,1})$, represented by an oriented Σ_g-bundle $\pi : E \to M$ over a closed oriented i-dimensional manifold M, satisfies the following three conditions:

(i) There is given an identification of the fibre E_{q_0} over a base point $q_0 \in M$ with Σ_g.

(ii) There is given a cross section $s : M \to E$ together with a trivialization $N(s(M)) \cong s(M) \times D^2$ of the normal disc bundle $N(s(M))$ of the image of the section s.

(iii) The image of the holonomy homomorphism $\pi_1(M) \to \mathcal{M}_{g,1}$ is contained in $\mathcal{K}_{g,1}$.

Henceforth let us call such a Σ_g-bundle a $\mathcal{K}_{g,1}$-*bundle*. Two $\mathcal{K}_{g,1}$-bundles $\pi_k : E_k \to M (k = 0, 1)$ over the same base space M are said to be isomorphic if there exists an isomorphism $E_0 \cong E_1$, over the identity of M, as Σ_g-bundles such that (i) the isotopy class of the induced diffeomorphism on $E_{q_0} = \Sigma_g$ is contained in $\mathcal{K}_{g,1}$ and (ii) it preserves the cross sections as well as the trivializations of their normal disc bundles. Then, as usual, there is a canonical bijection between the set of $\mathcal{K}_{g,1}$-isomorphism classes of $\mathcal{K}_{g,1}$-bundles over M and the set $[M, B\mathcal{K}_{g,1}]$ of homotopy classes of mappings from M to the classifying space $B\mathcal{K}_{g,1}$.

Now let $\pi : E \to M$ be a $\mathcal{K}_{g,1}$-bundle and let ξ be its tangent bundle along the fibres. Clearly we have an isomorphism

$$TE \cong \xi \oplus \pi^*(TM).$$

Since the image of the holonomy of π is contained in $\mathcal{K}_{g,1}$ by the assumption, the total space E contains $D^2 \times M$ as a natural submanifold. We write $E' = E \setminus \operatorname{Int} D^2 \times M$.

Proposition 7.2. *Let $\pi : E \to M$ be a $\mathcal{K}_{g,1}$-bundle. Then for each non-singular vector field X on Σ_g^0, there is a canonical framing t_X of ξ over E'. Moreover if we glue the bundle ξ along E' according to the above framing and pull it back to $(D^2/\partial D^2) \times M$ via the natural map $(D^2/\partial D^2) \times M \to E/E'$, then we obtain $\eta_{2-2g} \hat{\otimes} \varepsilon$ where η_{2-2g} is an oriented plane bundle over $D^2/\partial D^2 = S^2$ with the Euler number equal to $2-2g$, and ε denotes the trivial line bundle over M.*

Proof. First we make a cell decomposition of the base space M with a single 0-cell at the base point q_0. The non-singular vector field X defines a non-zero section of $\xi|_{E'_{q_0}}$ via the identification $E'_{q_0} = \Sigma_g^0$ where $E'_{q_0} = E_{q_0} \cap E'$. We extend this non-zero section to the whole of E' by an induction on the dimension of the cells. Let τ^1 be any 1-cell of M. Since the holonomy along this 1-cell is contained in $\mathcal{K}_{g,1}$ by the assumption, the non-zero section extends to $\pi^{-1}(\tau^1) \cap E'$ by Corollary 4.2. Here we can take the extension to be constant along $\pi^{-1}(\tau^1) \cap \partial E'$. Next let τ^2 be any 2-cell and let $\Sigma_g^0 \times D^2 \to \pi^{-1}(\tau^2) \cap E'$ be a trivialization over τ^2. By pulling back the non-zero section of ξ over $\pi^{-1}(\partial \tau^2) \cap E'$ via the above trivialization, we obtain a non-zero section σ^1 of ξ_0 (= the tangent bundle along the fibres of the trivial Σ_g^0-bundle $\Sigma_g^0 \times D^2$) over $\Sigma_g^0 \times \partial D^2$. This induces a mapping

$$h^1 : \Sigma_g^0 \times \partial D^2 \longrightarrow S^1$$

by the formula $h^1(p,q) = \angle(X_p, \sigma^1(p,q))$ $(p \in \Sigma_g^0, q \in \partial D^2)$. Here we consider X as a non-zero section of ξ_0 which is constant along the second factor of $\Sigma_g^0 \times D^2$. Observe that the above mapping h^1 is null homotopic. This is because the restriction of h^1 to $\Sigma_g^0 \times q_0$ (q_0 is the base point of ∂D^2 which is identified to the base point of M) and also to $\partial \Sigma_g^0 \times \partial D^2$ is the constant map to $0 \in S^1$. Hence there is an extension $\tilde{h}^1 : \Sigma_g^0 \times D^2 \to S^1$ of h^1. By making use of this extension, it is now easy to extend the non-zero section of ξ over each 2-cell τ^2.

Next let τ^3 be any 3-cell. Choose a trivialization

$$\Sigma_g^0 \times D^3 \to \pi^{-1}(\tau^3) \cap E'.$$

Then, as before, the pull back of the non-zero section obtained so far to the trivial bundle $\Sigma_g^0 \times D^3$ induces a mapping

$$h^2 : \Sigma_g^0 \times \partial D^3 \longrightarrow S^1,$$

which is easily shown to be null homotopic. Hence there is an extension $\tilde{h}^2 : \Sigma_g^0 \times D^3 \to S^1$ of h^2, and by making use of this we can extend the non-zero section over each 3-cell τ^3. Clearly we can continue the above argument over any k-cell ($k \geq 4$) so that we eventually obtain the required non-zero section of $\xi|_{E'}$. This induces our trivialization t_X of ξ over E'. Since the above non-zero section of ξ is constant along the second factor of $\partial E' = \partial D^2 \times M$, the latter statement of the Proposition is obvious. This completes the proof. q.e.d.

Let $\pi : E \to M$ be a $\mathcal{K}_{g,1}$-bundle and let t_X be the framing of ξ over E' as above. If the base space M is a closed oriented manifold of dimension $4i-3$ (so that $\dim E = 4i-1$), then we can define a rational number $\delta(\pi) \in \mathbf{Q}$ as follows. The definition is given by generalizing that of the Hirzebruch's signature defect to the context of *partially framed* $(4i-1)$-manifold (E, t_X).

First observe that all the Stiefel-Whitney numbers of the manifold E is zero so that we can choose a compact oriented $4i$-manifold W such that $\partial W = E$. Clearly we have $TW|_E = TE \oplus \varepsilon = \pi^*(TM) \oplus \xi \oplus \varepsilon$. In view of Proposition 7.2, the framing t_X of ξ over E' together with the unique trivialization of $\eta_{2-2g} \oplus \varepsilon$ (cf. the argument after Theorem 5.1) induces a canonical framing of the bundle $\xi \oplus \varepsilon$, which we denote also by t_X. Now choose any connection on TM. Then the pull back of this connection to $\pi^*(TM)$ together with the flat connection on $\xi \oplus \varepsilon$ corresponding to the framing t_X above defines a connection on $TW|_E$. Extend this connection to the whole of W. Now let $L_i = L_i(p_1, \cdots, p_i)$ be Hirzebruch's i-th L-polynomial ([13]). Then we can define the *relative* L_i-number $L_i(TW; t_X)$ with respect to the framing t_X on the boundary by

$$L_i(TW; t_X) = \int_W L_i(TW).$$

Here $L_i(TW)$ denotes the top form on W corresponding to the characteristic class L_i with respect to the connection on TW defined above. Now we set

$$\delta(\pi) = L_i(TW; t_X) - \operatorname{sign} W.$$

and call it the *signature defect* of the $\mathcal{K}_{g,1}$-bundle $\pi : E \to M$.

Proposition 7.3. *The above definition of the signature defect of a $\mathcal{K}_{g,1}$-bundle $\pi : E \to M$ is well defined. Namely it is independent of the choices of connections on TM and also of the manifold W such that $\partial W = E$.*

Proof. The proof can be given almost in the same way as the case of usual signature defect. It is enough to mention only the fact that any two connections on TM can be connected by a path of connections and that the possible error term, which is an integral over $E \times I$ corresponding to the difference of two connections on TM, vanishes because of the trivial dimension reasons. This completes the proof. q.e.d.

Next we show that our generalized signature defect of $\mathcal{K}_{g,1}$-bundles is a bordism invariant. More precisely we have

Proposition 7.4. *If two $\mathcal{K}_{g,1}$-bundles $\pi_k : E_k \to M_k$ $(k = 0, 1)$ over closed oriented $(4i-3)$-dimensional manifolds M_k are bordant, namely if they define the same element of $\Omega_{4i-3}(\mathcal{K}_{g,1})$, then their generalized signature defects coincide: $\delta(\pi_0) = \delta(\pi_1)$. It follows that the signature defect defines a homomorphism*

$$\delta_i : \Omega_{4i-3}(\mathcal{K}_{g,1}) \longrightarrow \mathbf{Q}.$$

Proof. By the assumption that the two $\mathcal{K}_{g,1}$-bundles π_0 and π_1 are bordant, there exists a $\mathcal{K}_{g,1}$-bundle $\tilde{\pi} : \widetilde{E} \to \widetilde{M}$ such that (i) $\partial\widetilde{M} = M_1 - M_0$ and $\tilde{\pi}|_{\partial \widetilde{E}} = \pi_1 - \pi_0$ and (ii) the identification of the fibre $\widetilde{E}_{\tilde{q}_0}$ of $\tilde{\pi}$ over the base point $\tilde{q}_0 \in \widetilde{M}$ induces those on the boundary. Let $\tilde{\xi}$ be the tangent bundle along the fibres of $\tilde{\pi}$. It is easy to see that the framing t_X given on the boundary can be extended to a framing $\tilde{t}_X$ of $\tilde{\xi}$ over $\widetilde{E}'$. Also any connections on TM_k can be extended to one on $T\widetilde{M}$. These induce a connection on $T\widetilde{E}$. But it is easy to see that the corresponding L_i-form on $\widetilde{E}$ vanishes identically. We know also that sign $\widetilde{E} = 0$ because the holonomy of this Σ_g-bundle is contained in $\mathcal{K}_{g,1}$. The required assertion follows from these facts easily. q.e.d.

Next we show that the signature defect of $\mathcal{K}_{g,1}$-bundles is invariant under the natural action of the mapping class group. Namely we have

Proposition 7.5. *The value of the signature defect of $\mathcal{K}_{g,1}$-bundles is independent of the choice of the non-singular vector field X on Σ_g^0. Hence the homomorphism $\delta_i : \Omega_{4i-3}(\mathcal{K}_{g,1}) \to \mathbf{Q}$ is $\mathcal{M}_{g,1}$-invariant where the mapping class group $\mathcal{M}_{g,1}$ acts on $\Omega_*(\mathcal{K}_{g,1})$ naturally.*

Proof. Let $\pi : E \to M$ be a $\mathcal{K}_{g,1}$-bundle over a closed oriented $(4i-3)$-dimensional manifold M. We have the partial framing t_X of TE corresponding to the non-singular vector field X. Namely it is a framing of the bundle $\xi \oplus \varepsilon$. Now let Y be another non-singular vector field on Σ_g^0

and let t_Y be the associated partial framing. Now consider the product bundle $\pi \times I : E \times I \to M \times I$. Choose any connection on TM. Since there is a canonical isomorphism $T(E \times I) = \pi^*(TM) \oplus \varepsilon \oplus (\xi \hat{\otimes} \varepsilon)$, we can consider the relative L_i-number $L_i(T(E \times I); t_Y - t_X)$. We claim that this number vanishes. To show this, it is enough to prove that with respect to the framings t_X and t_Y of $\varepsilon \oplus \xi$ on the boundary of $E \times I$, the relative p_1 of the bundle $\varepsilon \oplus (\xi \hat{\otimes} \varepsilon)$ is zero. Let us denote this relative class by $p_1(\varepsilon \oplus (\xi \hat{\otimes} \varepsilon); t_Y - t_X) \in H^4(E \times I, E \times \partial I; \mathbf{Q})$. Now we have also the relative Euler class $\chi \in H^2(E \times I, (E \times \partial I) \cup (D^2 \times M \times I); \mathbf{Z})$ of the bundle $\xi \hat{\otimes} \varepsilon$ with respect to the framings t_X and t_Y of $\xi|_{E'}$. Let us use the same symbol χ for the pull back of the above class to $H^2(E \times I, E \times \partial I; \mathbf{Z})$ which is isomorphic to $H^2(E; \mathbf{Z}) \oplus (H^1(E; \mathbf{Z}) \otimes H^1(I, \partial I; \mathbf{Z}))$. But clearly the restriction of χ to E is zero so that we can consider $\chi \in H^1(E; \mathbf{Z}) \otimes H^1(I, \partial I; \mathbf{Z})$. Hence $p_1(\varepsilon \oplus (\xi \hat{\otimes} \varepsilon); t_Y - t_X) = \chi^2 = 0$ as required. Thus we have proved that the relative L_i-number of $\pi \times I$ vanishes. We also have the trivial fact that sign $E \times I = 0$. It is now easy to deduce the required result from these two facts. The latter statement of the proposition is clear because the action of $\mathcal{M}_{g,1}$ on a $\mathcal{K}_{g,1}$-bundle is simply given by a change of the identification of the fibre over the base point with Σ_g so that its effect on the signature defect reduces to the change of the non-singular vector field. This completes the proof.

q.e.d.

As another general property of the homomorphisms δ_i, we prove that they are *stable* with respect to the genus. More precisely we have

Proposition 7.6. *Let $i : \mathcal{K}_{g,1} \to \mathcal{K}_{g+1,1}$ be the homomorphism induced from the natural embedding $\Sigma_g^0 \subset \Sigma_{g+1}^0$. Then the following diagram is commutative*

$$\begin{array}{ccc} \Omega_{4i-3}(\mathcal{K}_{g,1}) & \xrightarrow{\delta_i} & \mathbf{Q} \\ {\scriptstyle i_*}\downarrow & & \| \\ \Omega_{4i-3}(\mathcal{K}_{g+1,1}) & \xrightarrow{\delta_i} & \mathbf{Q}. \end{array}$$

Proof. Let $\pi : E \to M$ be a $\mathcal{K}_{g,1}$-bundle over a closed oriented $(4i-3)$-dimensional manifold M and let $u \in \Omega_{4i-3}(\mathcal{K}_{g,1})$ be its bordism class. As before we set $E' = E \setminus \operatorname{Int} D^2 \times M$ where $D^2 \times M$ is considered as a natural submanifold of E. Let $T_0 = T \setminus \operatorname{Int} D^2$ be the 2-torus with a hole. If we define $\hat{E}$ to be the manifold obtained by glueing

E' and $T_0 \times M$ along their boundaries, then the natural projection $\hat{\pi} : \hat{E} \to M$ admits a structure of a $\mathcal{K}_{g+1,1}$-bundle, and its bordism class in $\Omega_{4i-3}(\mathcal{K}_{g+1,1})$ is nothing but the element $i_*(u)$. Now let X be a non-singular vector field on Σ_g^0 and extend it to obtain a non-singular vector field $\hat{X}$ on Σ_{g+1}^0. Then we have partial framings t_X and $t_{\hat{X}}$ of the tangent bundles along the fibres of E and $\hat{E}$ respectively. Choose a compact oriented $4i$-dimensional manifold W such that $\partial W = E$. Let $\hat{T}$ be the solid torus and let $D^2 \subset \partial \hat{T}$ be an embedded disc so that $\partial \hat{T} \setminus \operatorname{Int} D^2 = T_0$. If we define $\hat{W}$ to be the manifold obtained by attaching $\hat{T} \times M$ to W along the "common" $D^2 \times M$ $(\subset \partial(\hat{T} \times M), E)$, then clearly we have $\partial \hat{W} = \hat{E}$. Now it is easy to see that $\operatorname{sign} \hat{W} = \operatorname{sign} W$ and $L_i(T\hat{W}; t_{\hat{X}}) = L_i(TW; t_X)$. Hence $\delta_i(i_*(u)) = \delta_i(u)$. This completes the proof. q.e.d.

Now we consider the value of the homomorphism

$$\delta_i : \Omega_{4i-3}(\mathcal{K}_{g,1}) \longrightarrow \mathbf{Q}$$

on the "decomposable" part

$$\sum_{p+q=4i-3, q>0} H_p(\mathcal{K}_{g,1}; \mathbf{Q}) \otimes \Omega_q \subset \Omega_{4i-3}(\mathcal{K}_{g,1}) \otimes \mathbf{Q}.$$

First it can be easily checked that the homomorphism

$$\delta_1 : \Omega_1(\mathcal{K}_{g,1}) = H_1(\mathcal{K}_{g,1}) \to \mathbf{Q}$$

coincides with the invariant $\frac{1}{3}d : \mathcal{K}_{g,1} \to \mathbf{Q}$. Let us write

$$d_1 \in H^1(\mathcal{K}_{g,1}; \mathbf{Q})$$

for $\delta_1 = \frac{1}{3}d$. Next consider the case of $i = 2$. Let $\pi : E \to S^1$ be a $\mathcal{K}_{g,1}$-bundle over S^1 and let N be a closed oriented 4-manifold. We can consider the product $\hat{\pi} : E \times N \to S^1 \times N$ as an element of

$$H_1(\mathcal{K}_{g,1}; \mathbf{Q}) \otimes \Omega_4 \subset \Omega_5(\mathcal{K}_{g,1}) \otimes \mathbf{Q},$$

so that we can consider the value $\delta_2(\hat{\pi}) \in \mathbf{Q}$. We claim that

$$\delta_2(\hat{\pi}) = d_1(\pi) L_1[N],$$

where $L_1[N]$ denotes the L_1-number of the 4-manifold N. To prove this, choose a compact oriented 4-manifold W such that $\partial W = E$.

Then clearly $\partial(W \times N) = E \times N$. Now we compute

$$\begin{aligned}\delta_2(\hat{\pi}) =& L_2(W \times N; \tilde{t}_X) - \operatorname{sign}(W \times N)\\ =& L_1(W; t_X) L_1[N] - \operatorname{sign} W \operatorname{sign} N\\ =& (L_1(W; t_X) - \operatorname{sign} W) L_1[N]\\ =& d_1(\pi) L_1[N],\end{aligned}$$

where t_X is the partial framing on TE, $\hat{t}_X$ denotes the induced partial framing on $T(E \times N)$ and we have used the fact that $L_1[N] = \operatorname{sign} N$ which is a consequence of the Hirzebruch's signature theorem. Now we consider the formal product $d_1 L_1$ as a generalized Pontrjagin number for the bordism group of $\mathcal{K}_{g,1}$. Namely for any $\mathcal{K}_{g,1}$-bundle $\omega : F \to M$ over a closed oriented 5-manifold M, we set $d_1 L_1(\omega) =< [M], d_1(\omega) L_1(M) >$ where $d_1(\omega) \in H^1(M; \mathbf{Q})$ is the d_1-class of the bundle ω. In view of the well known general theory (cf. [8]), this number induces a homomorphism

$$d_1 L_1 : \Omega_5(\mathcal{K}_{g,1}) \longrightarrow \mathbf{Q}.$$

In terms of this notation, it is easy to see that

$$d_1 L_1(\hat{\pi}) = d_1(\pi) L_1[N].$$

Set

$$d_2 = \delta_2 - d_1 L_1 : \Omega_5(\mathcal{K}_{g,1}) \longrightarrow \mathbf{Q}.$$

Then the above computation shows that d_2 is trivial on the decomposable summand

$$H_1(\mathcal{K}_{g,1}; \mathbf{Q}) \otimes \Omega_4 \subset \Omega_5(\mathcal{K}_{g,1}) \otimes \mathbf{Q}.$$

Hence we can consider d_2 as an element of $H^5(\mathcal{K}_{g,1}; \mathbf{Q})$ by Proposition 7.1.

Now we consider the homomorphism $\delta_i : \Omega_{4i-3}(\mathcal{K}_{g,1}) \to \mathbf{Q}$ for general i. Let $\pi : E \to M$ be a $\mathcal{K}_{g,1}$-bundle over a closed oriented $(4j-3)$-dimensional manifold M $(j < i)$ and let N be a closed oriented $4(i-j)$-dimensional manifold. Let $\hat{\pi} : E \times N \to M \times N$ be the product of π with N. Then almost the same proof as before implies the equality $\delta_i(\hat{\pi}) = \delta_j(\pi) L_{i-j}[N]$ where we use the Hirzebruch's signature theorem [13] $L_{i-j}[N] = \operatorname{sign} N$ in its full generality. Motivated by this, we set

$$d_i = \delta_i - d_{i-1} L_1 - d_{i-2} L_2 - \cdots - d_1 L_{i-1} : \Omega_{4i-3}(\mathcal{K}_{g,1}) \longrightarrow \mathbf{Q},$$

where $d_j L_k$ stands for the generalized Pontrjagin number corresponding to the formal product $d_j L_k$. Namely for a general $\mathcal{K}_{g,1}$-bundle $\omega : F \to Z$ over a closed oriented $(4j+4k-3)$-dimensional manifold Z, we define $d_j L_k(\omega) =< [Z], \rho^*(d_j) L_k(Z) >$ where $\rho : \pi_1(Z) \to \mathcal{K}_{g,1}$ is the holonomy homomorphism, and we understand $\rho^*(d_j)$ as an element of $H^{4j-3}(Z; \mathbf{Q})$. Then an easy inductive argument shows that d_i is trivial on the decomposable summand

$$\sum_{p+q=4i-3, q>0} H_p(\mathcal{K}_{g,1}; \mathbf{Q}) \otimes \Omega_q \subset \Omega_{4i-3}(\mathcal{K}_{g,1}) \otimes \mathbf{Q}.$$

Hence we can conclude that d_i is an element of $H^{4i-3}(\mathcal{K}_{g,1}; \mathbf{Q})$ by Proposition 7.1. In view of Proposition 7.5 and Proposition 7.6, we have established.

Theorem 7.7. *The generalized signature defect for $\mathcal{K}_{g,1}$-bundles defines a series of characteristic classes*

$$d_i \in H^{4i-3}(\mathcal{K}_{g,1}; \mathbf{Q})^{\mathcal{M}_{g,1}} \quad (i = 1, 2, \cdots),$$

which are $\mathcal{M}_{g,1}$-invariant and stable with respect to the genus g.

We call these cohomology classes the *secondary characteristic classes of surface bundles.*

8. Concluding remark

In this paper, we have formulated our results mainly at the level of the mapping class group $\mathcal{M}_{g,1}$ of Σ_g relative to an embedded disc $D^2 \subset \Sigma_g$ and its subgroup $\mathcal{K}_{g,1}$. However we expect that our results should descend to the level of the usual mapping class groups $\mathcal{M}_g, \mathcal{K}_g$. For example, we already have a formula for an explicit group 2-cocycle of $\mathcal{M}_g$ which is closely related to the intersection cocycle and represents the class $e_1 \in H^2(\mathcal{M}_g)$ (see [25]). The problem is that it is a *rational* cocycle rather than an integral one so that the associated mapping $\bar{d} : \mathcal{M}_g \to \mathbf{Q}$, which is an analogue of the invariant d for $\mathcal{M}_{g,1}$, is also rational valued. It would be interesting if one could give a geometrical interpretation to this invariant.

If we restrict $\bar{d}$ to the subgroup $\mathcal{K}_g$, then we obtain a homomorphism $\bar{d} : \mathcal{K}_g \to \mathbf{Q}$ which is $\mathcal{M}_g$-invariant. In fact we know that it is a generator of the group $H^1(\mathcal{K}_g; \mathbf{Q})^{\mathcal{M}_g}$ which is one dimensional (see [23]). Thus, after an appropriate adjustment, the secondary class $d_1 \in H^1(\mathcal{K}_{g,1}; \mathbf{Q})$

does descend to the one on the usual mapping class group. It seems to be very likely that higher secondary classes $d_i(i > 1)$ also descend to cohomology classes of $\mathcal{K}_g$.

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